An appendix to:

Zeros of equivariant vector fields: Algorithms for an invariant approach

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(November 29, 1993)

This appendix contains the results of several lengthy calculations which are mentioned in the main paper, *J. Symbolic Computation* (1994) **17**, 487-511.

A. Degenerate Linearization Conditions

In this appendix we examine the linearization of the normal form, Equation 8.2. As discussed in Section 6, the characteristic polynomial of the linearization at any equilibrium point is invariant under the group action, so the coefficients are polynomial invariants. We write the characteristic polynomial as

$$\mathrm{Det}(Df|_x - \lambda 1) = \lambda^4 + c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

and solve for the c_i in terms of the parameters and the fundamental invariants. To evaluate these expressions at the equilibrium points, we need to substitute in the calculated values of the invariants for that orbit of solutions. We do not perform that calculation since these expressions are already sufficiently complicated. Courtesy of Maple, we give you $c_0, \ldots, c_3$:

$$c_0 = -(6a + b + c + d)p_1 - 4l$$

$$\begin{aligned} c_1 = {} & 6l^2 + (3lb + 18la + 3lc + 3ld)p_1 + \\ & (3ab + \tfrac{dc}{2} - \tfrac{d^2}{2} + 2db - \tfrac{b^2}{2} - \tfrac{e^2}{2} - \tfrac{c^2}{2} + 3da + \tfrac{cb}{2})p_2 + \\ & (\tfrac{d^2}{2} + 12a^2 + 5ca - db + \tfrac{b^2}{2} + 2ab + \tfrac{e^2}{2} + \tfrac{c^2}{2} + 2da + \tfrac{cb}{2} + \tfrac{dc}{2})p_1^2 + \\ & (6ab - 2e^2 - 4c^2 + 6da - 12ca + dc + cb + 2db)q_1 \end{aligned}$$

$$
\begin{aligned}
c_2 = {} & (-3l^2b - 3dl^2 - 18l^2a - 3l^2c)p_1 - 4l^3 + \\
& (-4bdl - 6dla - bcl - 6bla + e^2l - dcl + c^2l + d^2l + b^2l)p_2 + \\
& (-bcl - 4bla - e^2l - dcl - c^2l - 4dla - d^2l - 24la^2 - \\
& b^2l + 2bdl - 10cla)p_1^2 + \\
& (4e^2l - 12bla - 2bcl - 2dcl + 8c^2l - 12dla - 4bdl + 24cla)q_1 + \\
& (c^3 - \tfrac{b^3}{3} - \tfrac{d^3}{3} - \tfrac{8b^2c}{3} + \tfrac{2e^2b}{3} - 2d^2c - bd^2 - 12dca - 12acb - \tfrac{10dcb}{3} - \\
& \tfrac{2c^2b}{3} + 6c^2a - \tfrac{4c^2d}{3} + \tfrac{5b^2d}{3} + 2e^2a + \tfrac{e^2c}{3} - 4d^2a + 4dab)p_3 + \\
& (\tfrac{b^3}{2} - \tfrac{3c^3}{2} + \tfrac{d^3}{2} + 4b^2c - \tfrac{e^2b}{2} - 6da^2 + \tfrac{7d^2c}{2} + \\
& 13dca + 13acb + 2dcb + \tfrac{3c^2b}{2} - 7c^2a + 2c^2d + 2b^2a - 2b^2d - 6a^2b - \\
& e^2a - \tfrac{e^2c}{2} + 5d^2a - 14dab)p_1p_2 + \\
& (\tfrac{c^3}{2} - \tfrac{b^3}{6} - \tfrac{d^3}{6} - \tfrac{4b^2c}{3} - \tfrac{e^2b}{6} - da^2 - \tfrac{3d^2c}{2} - e^2a + bd^2 - \\
& 5dca - 5acb + \tfrac{dcb}{3} - 10a^3 - \tfrac{5c^2b}{6} + c^2a - \tfrac{2c^2d}{3} - 2b^2a + \tfrac{b^2d}{3} - \\
& a^2b - 7a^2c + \tfrac{e^2c}{6} - d^2a + 6dab)p_1^3 + \\
& (e^2b - b^2c - 12da^2 - 3bd^2 + 2dca + 2acb - 4dcb + 4c^2b + 10c^2a + \\
& 3c^2d + b^2d - 12a^2b + 24a^2c + 8e^2a + e^2c - 6d^2a - 8dab)p_1q_1 + \\
& (e^2b - b^2c - c^2d + d^2c - 4bd^2 + c^2b - 6d^2a + 6b^2a + 4b^2d - e^2d)q_2 + \\
& (4d^2e + 4eb^2 + 4ced - 12aec - 4c^2e + 12abe + 4ebc - 2e^3 + 12aed)q_3
\end{aligned}
$$

$$
\begin{aligned}
c_3 = {} & l^4 + (dl^3 + cl^3 + l^3b + 6l^3a)p_1 + \\
& (3al^2b + 2dl^2b + \tfrac{dcl^2}{2} - \tfrac{b^2l^2}{2} - \tfrac{d^2l^2}{2} - \tfrac{c^2l^2}{2} - \tfrac{e^2l^2}{2} + \tfrac{l^2cb}{2} + 3dl^2a)p_2 + \\
& (12a^2l^2 + \tfrac{b^2l^2}{2} + \tfrac{d^2l^2}{2} + 2al^2b + \tfrac{c^2l^2}{2} + \tfrac{e^2l^2}{2} + 5l^2ca + \\
& \tfrac{l^2cb}{2} - dl^2b + 2dl^2a + \tfrac{dcl^2}{2})p_1^2 + \\
& (-4c^2l^2 - 12l^2ca + 6dl^2a + 6al^2b - 2e^2l^2 + l^2cb + 2dl^2b + dcl^2)q_1 + \\
& (12lacb + 12ldca + \tfrac{10ldcb}{3} - 4ldab - 2e^2la - \tfrac{2e^2bl}{3} - lc^3 - \tfrac{5ldb^2}{3} + \tfrac{4ldc^2}{3} + \tfrac{ld^3}{3} + \\
& 2ld^2c + \tfrac{8lcb^2}{3} - 6lac^2 + 4lad^2 + lbd^2 + \tfrac{2lc^2b}{3} + \tfrac{lb^3}{3} - \tfrac{le^2c}{3})p_3 + \\
& (6la^2b + 6lda^2 - 13lacb - 13ldca - 2ldcb + 14ldab + e^2la + \tfrac{e^2bl}{2} + \\
& \tfrac{3lc^3}{2} + 2ldb^2 - 2ldc^2 - \tfrac{ld^3}{2} - \tfrac{7ld^2c}{2} - 4lcb^2 + 7lac^2 - \\
& 5lad^2 - 2lab^2 - \tfrac{3lc^2b}{2} - \tfrac{lb^3}{2} + \tfrac{le^2c}{2})p_1p_2 + \\
& (la^2b + lda^2 + 5lacb + 5ldca - \tfrac{ldcb}{3} + 7la^2c - 6ldab + 10la^3 + e^2la + \\
& \tfrac{e^2bl}{6} - \tfrac{lc^3}{2} - \tfrac{ldb^2}{3} + \tfrac{2ldc^2}{3} + \tfrac{ld^3}{6} + \tfrac{3ld^2c}{2} + \tfrac{4lcb^2}{3} - \\
& lac^2 + lad^2 + 2lab^2 - lbd^2 + \tfrac{5lc^2b}{6} + \tfrac{lb^3}{6} - \tfrac{le^2c}{6})p_1^3 + \\
& (12la^2b + 12lda^2 - 2lacb - 2ldca + 4ldcb - 24la^2c + 8ldab - 8e^2la - e^2bl - \\
& ldb^2 - 3ldc^2 + lcb^2 - 10lac^2 + 6lad^2 + 3lbd^2 - 4lc^2b - le^2c)p_1q_1 + \\
& (ldc^2 + e^2ld - e^2bl - 4ldb^2 - 6lab^2 - ld^2c + lcb^2 + 6lad^2 + 4lbd^2 - lc^2b)q_2 + \\
& (12laec - 4eb^2l - 12aedl - 4edcl - 12aelb + 2e^3l + 4c^2el - 4cebl - 4eld^2)q_3 + \\
& (48adcb + 3e^2bd - 3e^2cd + 18ab^3 - 6adb^2 - 12ac^3 + 24ad^3 + 6ad^2c + 12ac^2d - \\
& 18b^2ac + \tfrac{3e^4}{4} + 6bac^2 - 24ad^2b - \tfrac{3e^2b^2}{2} - \tfrac{3e^2d^2}{2} + \tfrac{3e^2c^2}{2} - 6e^2ad + \tfrac{3b^2c^2}{2} + \\
& 15bc^2d - \tfrac{9b^2d^2}{2} + 3bd^2c + 6b^2cd + \tfrac{15d^4}{4} + 9d^3c + \tfrac{15b^4}{4} - \tfrac{9c^4}{4} - 3c^3b + \tfrac{3c^2d^2}{2})p_4 + \\
& (12a^2dc - 54adcb + 12a^2cb - 4a^2db - 3e^2bd - 2e^2a^2 + 3e^2cd - 23ab^3 + 9adb^2 + \\
& 15ac^3 - 29ad^3 + 4ad^2c - 6a^2c^2 - 14ac^2d + 26b^2ac - e^4 - 10bac^2 + 4a^2d^2 + \\
& 31ad^2b - 3e^2ac + 2e^2b^2 + 2e^2d^2 - 3e^2c^2 + 8e^2ad - b^2c^2 - 19bc^2d + 7b^2d^2 - \\
& 2bd^2c - 7b^2cd - 5d^4 - 9d^3c - 5b^4 + 3c^4 + 3c^3b - c^2d^2)p_1p_3 +
\end{aligned}
$$

$(-21adcb-\frac{9e^2bd}{4}-\frac{3e^2ba}{2}+\frac{3e^2cd}{2}-9ab^3+\frac{3adb^2}{2}+6ac^3-12ad^3-\frac{3ad^2c}{2}-$
$\frac{15ac^2d}{2}+\frac{21b^2ac}{2}-\frac{3e^4}{8}-\frac{9bac^2}{2}+\frac{21ad^2b}{2}+\frac{3e^2b^2}{4}+\frac{3e^2d^2}{4}-\frac{3e^2c^2}{4}+\frac{3e^2ad}{2}-\frac{3b^2c^2}{4}-$
$\frac{33bc^2d}{4}+\frac{3b^2d^2}{2}-\frac{3bd^2c}{4}-\frac{9b^2cd}{4}-\frac{15d^4}{8}-\frac{9d^3c}{2}-\frac{15b^4}{8}+\frac{9c^4}{8}+\frac{3c^3b}{2}-\frac{3c^2d^2}{4})p_2^2+$
$(3a^3d+36adcb-\frac{27a^2dc}{2}-\frac{27a^2cb}{2}+12a^2db+3a^3b+3e^2bd+\frac{3e^2ba}{2}+\frac{3e^2a^2}{2}-$
$\frac{3e^2cd}{2}+\frac{33ab^3}{2}-6adb^2-\frac{21ac^3}{2}+\frac{39ad^3}{2}-\frac{3a^2b^2}{2}-\frac{27ad^2c}{2}+\frac{15a^2c^2}{2}+12ac^2d-$
$24b^2ac+\frac{3e^4}{4}+\frac{21bac^2}{2}-\frac{9a^2d^2}{2}-18ad^2b+\frac{9e^2ac}{2}-\frac{3e^2b^2}{2}-\frac{3e^2d^2}{2}+3e^2c^2-$
$3e^2ad+15bc^2d-\frac{9b^2d^2}{2}+3b^2cd+\frac{15d^4}{4}+\frac{9d^3c}{2}+\frac{15b^4}{4}-\frac{9c^4}{4}-\frac{3c^3b}{2})p_1^2p_2+$
$(\frac{9a^2dc}{2}-6adcb+\frac{9a^2cb}{2}-5a^2db+3a^3c+3a^4-\frac{3e^2bd}{4}+\frac{e^2a^2}{2}-\frac{5ab^3}{2}+\frac{3adb^2}{2}+$
$\frac{3ac^3}{2}-\frac{5ad^3}{2}+\frac{3a^2b^2}{2}+5ad^2c-\frac{3a^2c^2}{2}-\frac{5ac^2d}{2}+\frac{11b^2ac}{2}-\frac{e^4}{8}-2bac^2+\frac{a^2d^2}{2}+$
$\frac{ad^2b}{2}-\frac{3e^2ac}{2}+\frac{e^2b^2}{4}+\frac{e^2d^2}{4}-\frac{3e^2c^2}{4}-\frac{e^2ad}{2}+\frac{b^2c^2}{4}-\frac{11bc^2d}{4}+\frac{b^2d^2}{2}-\frac{bd^2c}{4}+$
$\frac{b^2cd}{4}-\frac{5d^4}{8}-\frac{5b^4}{8}+\frac{3c^4}{8}+\frac{c^2d^2}{4})p_1^4+$
$(\frac{9e^2bd}{2}-18adcb+9e^2ba+3e^2cd-6ab^3-9adb^2-6ac^3-12ad^3-15ad^2c+$
$21ac^2d+9b^2ac+27bac^2+9ad^2b+3e^2b^2+3e^2d^2+\frac{3e^2c^2}{2}+15e^2ad+\frac{15b^2c^2}{2}+$
$\frac{3bc^2d}{2}+\frac{3b^2d^2}{2}-\frac{3bd^2c}{2}-3b^3d-\frac{9b^2cd}{2}-9d^3c-3bd^3+3c^3b+\frac{15c^2d^2}{2})p_2q_1+$
$(6a^3d+6adcb-3a^2dc-3a^2cb+6a^2db+6a^3b-12a^3c+\frac{3bc^2d}{2}+\frac{3e^2bd}{2}-$
$6e^2a^2-3ad^2c-6a^2c^2-6b^2ac-3bac^2+6a^2d^2+6ad^2b+3e^2ac+\frac{3e^2c^2}{2}+$
$3e^2ad-\frac{3b^2c^2}{2}-\frac{3c^2d^2}{2}+\frac{3b^2d^2}{2}+\frac{3bd^2c}{2}-\frac{3b^2cd}{2})p_1^2q_1+$
$(3ad^2c-3e^2ba-6adb^2-6a^2b^2+6a^2d^2+3ac^2d-3b^2ac-3bac^2-3b^2cd+$
$6ad^2b+3e^2ad+3bd^2c)p_1q_2+$
$(12debc-18c^2ea-12a^2eb-12a^2ed+12aecd-6d^2ea-6c^3e+12a^2ec-$
$6eb^2a+12aecb)p_1q_3+$
$(12aecb+6d^3e-24eb^2a+6bd^2e+6c^2eb-6eb^3-6b^2de+24d^2ea-$
$6dec^2-12aecd)q_4+$
$(12adcb-6e^2bd-12e^2ba-3e^2cd+6ab^3+6adb^2+6ac^3+12ad^3+24ad^2c-$
$24ac^2d-30bac^2-12ad^2b-6e^2ac-3e^2b^2-3e^2d^2-3e^2c^2-18e^2ad-6b^2c^2-$
$6bc^2d-3b^2d^2+3bd^2c+3b^3d+6b^2cd+9d^3c+3bd^3-3c^3b-6c^2d^2)q_5+$
$(24aecb+6eb^2c-30c^2ea+12debc+24aecd+6d^2ea-6c^3e+6eb^2a+$
$6d^2ec-12adbe-6e^3a)q_6+$
$(3bd^2c-6e^2ba-3e^2cb+3e^2cd+6ab^3-18adb^2-6ad^3-24ad^2c-6ac^2d+$
$24b^2ac+6bac^2+18ad^2b+6e^2ad+3c^3b+9b^3c-3b^2cd-9d^3c-3c^3d)q_7\ .$

We also present the expression for $s(e)$ from Section 8, Equation 8.3.

$$
\begin{aligned}
s(e) = {} & e^{12} + (8db - 6d^2 - 6b^2 - 4dc - 4cb + 6c^2)e^{10} + \\
& (16b^3c - 40b^3d - 16c^2d^2 + 15b^4 - 16b^2c^2 + 40bc^2d + 15c^4 - 20dc^3 - \\
& 8d^2cb + 15d^4 - 40d^3b - 20c^3b + 52b^2d^2 - 8db^2c + 16cd^3)e^8 + \\
& (108d^2c^2b^2 + 12d^4c^2 - 20b^6 - 20d^6 + 44b^4dc - 20b^3d^2c - 20d^3b^2c + 12c^2b^4 - \\
& 72c^2b^3d + 44cbd^4 + 80b^5d - 148b^4d^2 + 176d^3b^3 + 36d^3c^3 + 36c^3b^3 - 148d^4b^2 + \\
& 80d^5b - 72d^3bc^2 - 24cb^5 - 28c^3bd^2 - 24cd^5 - 28b^2c^3d + 80dc^4b - 40c^5b - \\
& 4c^4b^2 + 20c^6 - 40c^5d - 4c^4d^2)e^6 + \\
& (72d^3c^2b^3 - 20d^3c^3b^2 - 58d^2c^2b^4 + 32c^3b^4d + 20c^2b^5d + 20d^5c^2b - 58d^4c^2b^2 - \\
& 12d^5c^3 - 12c^3b^5 + 2d^6c^2 + 12c^5d^3 + 2c^2b^6 - 20d^2c^3b^3 + 32d^4c^3b + 16cb^7 - \\
& 6c^4b^4 + 12c^5b^3 - 6c^4d^4 + 24c^6b^2 + 16d^7c + 15b^8 - 8d^4cb^3 + 36d^5cb^2 + \\
& 36cb^5d^2 - 44dcb^6 - 8d^3cb^4 - 36c^5b^2d - 24c^4db^3 - 24c^4bd^3 - 36c^5d^2b + \\
& 48c^4b^2d^2 + 192b^6d^2 + 322b^4d^4 + 192d^6b^2 + 80c^6db - 40c^7d + 24c^6d^2 - 40c^7b \\
& + 15c^8 + 15d^8 - 80db^7 - 44d^6bc - 288d^5b^3 - 288b^5d^3 - 80d^7b)e^4 + \\
& (26c^8b^2 - 20c^9d - 20c^9b + 6c^{10} - 6d^{10} - 4cb^9 + 8dc^6b^3 + 12d^2c^5b^3 - \\
& 12dc^5b^4 + 4d^2c^4b^4 - 20d^2c^6b^2 + 12d^3c^5b^2 + 4d^4c^4b^2 - 20d^2c^7b + 8d^3c^6b - \\
& 12d^4c^5b + 12c^3d^6b - 12c^3d^2b^5 + 4d^3c^3b^4 - 12d^5c^3b^2 + 4d^4c^3b^3 - 6d^8c^2 - \\
& 4c^4d^6 - 68d^2c^2b^6 - 68d^3cb^6 + 44d^4cb^5 + 44d^5cb^4 - 68d^6cb^3 + 12dc^3b^6 + \\
& 152d^3c^2b^5 + 152d^5c^2b^3 - 204d^4c^2b^4 + 24d^7c^2b - 68d^6c^2b^2 - 4b^7c^3 - 20c^7b^3 + \\
& 12c^6b^4 + 20d^7cb^2 + 24b^7c^2d + 20b^7cd^2 - 4c^3d^7 - 4b^6c^4 - 6b^8c^2 + 8cd^8b + \\
& 40db^9 + 8b^8cd + 40d^9b + 12c^6d^4 - 20dc^7b^2 - 4d^9c + 40dbc^8 - 6b^{10} - \\
& 118d^8b^2 - 118b^8d^2 - 260b^6d^4 - 260b^4d^6 + 272b^5d^5 + 208b^7d^3 + 208d^7b^3 - \\
& 20c^7d^3 + 26c^8d^2)e^2 + \\
& (10c^6d^6 - 4c^5b^7 - 4d^7c^5 - 12c^7d^5 + 15c^8b^4 + 10c^6b^6 - 12c^7b^5 + 15c^8d^4 + \\
& 4db^{10}c - 4db^2c^9 + 8dbc^{10} - 8db^{11} - 8db^7c^4 - 12db^9c^2 - 12db^5c^6 + 4db^8c^3 + \\
& b^{12} - 4c^{11}b + 2d^{10}c^2 + 2b^{10}c^2 - 4d^2bc^9 - 20d^2b^7c^3 + 32d^2b^6c^4 + 6d^2b^4c^6 + \\
& 24d^2b^5c^5 - 28d^2b^9c + 12d^2b^3c^7 + 3d^8c^4 + 28d^2b^{10} + 42d^2b^8c^2 - 12d^2c^8b^2 + \\
& 8c^{10}d^2 + c^{12} - 4) \, .
\end{aligned}
$$