# Master-slave synchronization from the point of view of global dynamics

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With an appendix by Hyman Bass [2]

**Abstract** We present a mathematical framework for the theory of a synchronization phenomenon for dynamical systems discovered by Pecora & Carroll. From this perspective, we can synchronize, using a single coordinate, an open dense set of linear systems. We use our insights to synchronize nonlinear systems which were not previously recognized as being synchronizable.

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# 1 Introduction

The common meaning of synchronization is the concurrence of events or motions with respect to time. Synchronization, in its most current use in physics, refers to systems of coupled identical systems, where the coupling can be intrinsic or due to a common forcing (*e.g.*, [5, 1]). We do not consider this kind of synchronization in the current paper. As a physical phenomenon, it might be tempting to call the frequency locking (to identical frequencies) of coupled oscillators synchronization. This is however too imprecise because the phase has an obvious role to play in synchronization. We wish to keep the notion of synchronization in scientific considerations the same as its common usage. Recently, L. M. Pecora and T. L. Carroll have presented a new phenomenon where a system synchronizes with another one with coupling active in only one direction, a master-slave type of synchronization. This is the phenomenon which we consider and in this paper synchronization will always stand for master-slave synchronization.

Let

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

be the evolution equations of some dynamical system, where $x, y$ are respectively $m$ and $n$-dimensional vectors and where the prime stands either for differentiation with respect to time (in the case of a vector field) or for the next values in the case of a map. With $t$ standing for either continuous or discrete time, and $(x(t), y(t))$ standing for the solutions of (1) with initial conditions $(x_0, y_0)$, let us define the $(x_0, y_0)$-$y$-*subordinate system* to (1) as

$$Y' = g(x(t), Y) \ . \tag{2}$$

If $Y(t) - y(t)$ converges to 0 as $t \to \infty$ for every $(x_0, y_0)$ and every initial condition on $Y$, we say that $x$ is a *synchronizing coordinate* for (1). This is merely a formalization of ideas introduced by Pecora and Carroll in [10]. They notice in particular that for the Lorenz equations [8]

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{3}$$

in some range of the parameter space, both $x$ and $y$ are synchronizing coordinates, while $z$ is not. Furthermore, they describe necessary conditions for synchronization for the system (2) and show that $x$ and $y$ are indeed synchronizing for (3) in some parameter range. This work is described in a series of papers [10, 11] and has been applied to masking signals in view of secret transmissions in another series of papers [2, 3].

We first outline the approach presented by Pecora and Carroll. Consider how the subordinate system in (2) evolves with respect to the initial system (1) by defining

$\Delta y = Y - y$ and then

$$\begin{aligned} \Delta y' &= g(x(t), y(t) + \Delta y) - g(x(t), y(t)) \\ &= D_y g(x(t), y(t))\Delta y + o(\Delta y) \end{aligned} \tag{4}$$

Pecora and Carroll note that the Lyapunov exponents for the evolution of this system, which have been called *conditional Lyapunov exponents* since they depend on a trajectory in the original system, determine the local stability of a solution. The existence of a positive conditional Lyapunov exponent defies synchronization. To get global synchronization results, other techniques are required. This may be done in some simple cases; for the Lorenz equations a Lyapunov function is produced [6, 2, 3].

All these considerations are expressed in terms of equations describing dynamical systems. The aim of this paper is to re-examine the theory of synchronization from the perspective of global dynamical systems theory, and, more specifically, to describe what one can gain in the comprehension of synchronization from the point of view of the intrinsic definition of dynamical systems and from geometric considerations.

This paper is organized as follows. In Section 2 we give a number of simple examples of synchronizing dynamical systems which illustrate the basic ideas. In Section 3 we present our general theory of synchronization. This discussion allows us to find synchronizing coordinates in cases where none were previously available. This is followed by a discussion of synchronization for linear problems in Section 4. Finally, in Section 5, we reconsider the Lorenz equations and also discuss the standard map, the Hénon map, a three-dimensional linear map, and the Rössler equations. In the Appendix we include some work of Hyman Bass which we use in Section 4. This result essentially indicates that the eigenvalues of a submatrix of a matrix may be arbitrary and unrelated to the eigenvalues of the whole matrix. Subsequent to the initial writing of this paper, Alan Hoffman drew our attention to the work of Farahat and Ledermann [4] from which this result also follows. With the exception of the Appendix, we state all our theorems without proofs because the proofs follow almost directly from the definitions and previous theorems.

We notice that synchronization theory has been anchored into physics by several experimental works, *e.g.* [9, 13], but we do not here examine the direct relevance of our work to physical systems.

## 2 Examples

We will present a number of examples of simple dynamical systems and consider them from the viewpoint of synchronization.

**The solenoid**

The solenoid is the unique attractor of the map $f$ from the solid torus $\mathcal{T} = \mathbb{T}^1 \times \mathbb{D}^2$

to itself, defined by

$$f(\theta, x, y) = (2\theta, \epsilon_1 \cos\theta + \epsilon_2 x, \epsilon_1 \sin\theta + \epsilon_2 y)\ , \tag{5}$$

where $0 < \epsilon_2 < \epsilon_1 < 1 - \epsilon_2$.

For any chosen initial condition $P_0 = (\theta_0, x_0, y_0)$ in $\mathcal{T}$, with forwards orbit $P_n = (\theta_n, x_n, y_n)$, the $P_0$-$(x,y)$-subordinate system is

$$(X_{n+1}, Y_{n+1}) = (\epsilon_1 \cos\theta_n + \epsilon_2 X_n, \epsilon_1 \sin\theta_n + \epsilon_2 Y_n) \tag{6}$$

so that initial discrepancies in the $(x,y)$-coordinates diminish at rate $\epsilon_2$ and the coordinate $\theta$ is synchronizing. In terms of dynamical systems theory, this can be understood from the fact that $(X_n, Y_n)$ are the $(x,y)$ coordinates of the $n^{th}$ iterate of $(\theta_0, X_0, Y_0)$ under $f$ and the fact that the disk $\theta = \theta_0$ of $\mathcal{T}$ is part of the stable manifold of the orbit of $P_0$, for any $P_0$ in that disk.

**A linear map**

Consider the linear map on $\mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\ . \tag{7}$$

Neither $x$ nor $y$ is a synchronizing coordinate, but from the examination of the previous example, we can rewrite (7) in the basis made from the stable and unstable directions. The equation would then read

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}' = \begin{pmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}\ , \tag{8}$$

where $|\lambda_s| < 1 < \lambda_u = -1/\lambda_s$ are the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. It is clear that $\xi$ is a synchronizing coordinate, but simpler handling of (7), not any more in the line of dynamical systems ideas, can yield even better results.

If we perform a simple change of coordinates given by rotation by $\alpha$, the new mapping is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 + \frac{3}{2}\sin 2\alpha & 1 - 3\sin^2\alpha \\ 3\cos^2\alpha - 1 & 1 - \frac{3}{2}\sin 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\ . \tag{9}$$

Synchronizing in $x$ now results in the difference $\Delta y \equiv y_1 - y$ evolving by $\Delta y' = (1 - \frac{3}{2}\sin 2\alpha)\Delta y$. Alternatively, synchronizing with $y$ gives $\Delta x' = (1 + \frac{3}{2}\sin 2\alpha)\Delta x$. So for any rotated coordinate system different from the original one, there is always a synchronizing subsystem. Furthermore, in this simple linear problem, coordinates may be chosen ($\sin 2\alpha = \frac{2}{3}$) so that the evolution of $y$ depends only on $x$ thus allowing perfect synchrony in one step.

**An Anosov diffeomorphism**

Consider the following Anosov diffeomorphism

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 , \tag{10}$$

defined on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We apply the change of coordinates given by

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 , \tag{11}$$

to get the same dynamical system expressed as

$$\begin{pmatrix} z \\ w \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \mod 1 . \tag{12}$$

In these new coordinates, $z$ is a synchronizing coordinate since the evolution of $w$ does not depend on $w$.

**The Lorenz equations**

The Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{13}$$

(with $\sigma, r, b > 0$) is the example most discussed by Pecora and Carroll. We have already reported on their methods and findings, so we shall here only add a couple of remarks:

1. Pecora and Carroll observe that the $z$-coordinate is not synchronizing while the $x$ and $y$ coordinates are. Due to the symmetry of the equations, for any orbit which lies off the axis of symmetry (the $z$-axis) there will be a second orbit whose projection onto the $z$-axis will be identical to that of the original. Thus, using the $z$-coordinate to synchronize to a trajectory which is not asymptotic to the z-axis will not work.

2. The Lorenz system has a simple structure which results in a linear subsystem being produced when either $x$ or $y$ is used as a synchronizing coordinate and also allows for the production of a Lyapunov function with which to prove synchronization. It is also easy to see that any orbit which lies off the axis of symmetry is uniquely determined from the projection onto its $x$-coordinate. This follows from the observation that if you know $x(t)$ then you know $\dot{x}(t)$ and can use the differential equation to compute $y(t)$. You then know $\dot{y}(t)$ and can compute $z(t)$ from the differential equation.

Although on the $z$-axis, an orbit is not uniquely determined by it's other components (which are identically 0), all trajectories converge to the origin, so the flow is synchronizing.

3. On the strange attractor (for the parameter regime in which it exists), there is one negative Lyapunov exponent, one zero Lyapunov exponent, and one positive Lyapunov exponent, in contrast to one positive Lyapunov exponent and two negative ones in the case of the solenoid. So Pecora and Carroll's observations on synchronization cannot all be captured by stable manifold theory like in the simplest case of the solenoid.

Let us make some final observations regarding the examples of this section.

1. In each case, the fact that the synchronizing variable comes from an actual orbit of the original dynamical system is irrelevant for the subordinate variables to converge to an orbit independent of the initial conditions. This will motivate us to define in the following section a more general notion of synchronization than the one given by Pecora and Carroll.

2. The linear example demonstrated that the original formulation of synchronization is too dependent on the equations, instead of the intrinsic dynamical system considered as an object before a coordinate system is chosen. The treatment given here for the linear map and Anosov diffeomorphism only involves a linear change of coordinates. In general, nonlinear changes of variables may also need to be considered.

# 3 The general theory of master-slave synchronization

In this section we will place the theory of synchronization into a global dynamical systems framework. We will first do this for mappings, and then we will present the analogous setup for vector fields.

**Synchronization for mappings**

Let $\mathcal{M}$ be a manifold of dimension greater than one and let $F : \mathcal{M} \to \mathcal{M}$ be a mapping. Assume that $\mathcal{M}$ has a product structure, so that $\mathcal{M} = M \times N$ with $M, N$ manifolds. For any choice of product structure, we can perform the following construction to produce a nonautonomous mapping on one of the factors, say $N$.

Let $\{x(i)\}_{i=0}^{\infty}$ be a sequence of points in $M$. Define $\tilde{F}_x : \mathbb{N} \times N \to N$ by

$$\tilde{F}_x(n, y) = \Pi_N F(x(n), y) \ , \tag{14}$$

where $\Pi_N$ is the projection of $\mathcal{M}$ onto its second factor $N$. A nonautonomous mapping on $N$ is defined by

$$Y(n+1) = \tilde{F}_x(n, Y(n)) \; . \tag{15}$$

The trajectory on $N$ may be lifted to a motion in $\mathcal{M}$ by considering for each $n \in \mathbb{N}$, the point $(x(n), Y(n)) \in \mathcal{M}$. This gives us, for each $n$, a mapping of the submanifold $\{x(n)\} \times N$ to the submanifold $\{x(n+1)\} \times N$.

Based on the behavior of the dynamical system $\tilde{F}_x$ on $N$, we can define several notions of synchrony. We assume that there is a metric on $N$, $d_N(\cdot,\cdot)$, that is used to define convergence of trajectories of (15).

**Definition 1** *A system $(\mathcal{M}, F, M, N)$ is called* absolutely synchronizing *if for all sequences $\{x(i)\}$, the motion under $\tilde{F}_x$ on $N$ of all initial conditions converges, i.e., for all $Y_1(0), Y_2(0)$, $\lim_{n\to\infty} d_N(Y_1(n), Y_2(n)) = 0$.*

**Definition 2** *A system $(\mathcal{M}, F, M, N)$ is called* synchronizing *if for all sequences $\{x(i)\}_{i=0}^{\infty}$ which are the sequence of first factors of a trajectory on $\mathcal{M}$ under $F$, the motion on $N$ converges.*

**Definition 3** *A system $(\mathcal{M}, F, M, N)$ is called* $\{x(i)\}$-conditionally synchronizing *for a specific sequence $\{x(i)\}$, if the motion on $N$ induced by that sequence converges.*

These definitions have all been global in terms of the initial conditions allowed. At times, it may also be useful to consider local versions of these definitions.

Finally, we want to define what it means to be able to synchronize a dynamical system. It is the notion of being able to find a product structure such that the resulting system is synchronizing.

**Definition 4** *A system $(\mathcal{M}, F)$ is called* synchronizable with $k$ inputs *if there exists a product structure $\mathcal{M} = M \times N$ with dim$(M) = k$ such that $(\mathcal{M}, F, M, N)$ is synchronizing.*

We summarize the essential idea here. We would like to find a product structure on $\mathcal{M}$ such that if $(x(n), y(n))$ is a trajectory of $F$, then there is another motion $Y(n)$ derived from $\tilde{F}_x$ and an initial condition in $N$. If we consider this as a motion on $N$, then we would like $\lim_{n\to\infty} d_N(Y(n), y(n)) = 0$. Alternatively, we may lift the motion $Y(n)$ to $\mathcal{M}$ to get $(x(n), Y(n))$ and we would like to see if this converges to $(x(n), y(n))$. If this happens for all initial conditions $x(0), y(0), Y(0)$, then the selected product structure is synchronizing for the dynamical system. (Note that if $Y(0) = y(0)$, then $Y(n) = y(n)$.) As seen by the linear map example of Section 2, the property of synchronizing is highly dependent on the choice of product structure.

**Synchronization for vector fields**

We now present the analogous setup for vector fields. Let $\mathcal{M}$ be a manifold and $F$ a vector field on $\mathcal{M}$. Assume that $\mathcal{M}$ has a product structure, so that $\mathcal{M} = M \times N$, and for any choice of product structure we construct a nonautonomous vector field on $N$.

Let $x : \mathbb{R} \to M$ be given. Define $\tilde{F}_x : \mathbb{R} \times N \to TN$ by

$$\tilde{F}_x(t, y) = \Pi_N F(x(t), y) \ , \tag{16}$$

where $\Pi_N : T_{(x,y)}\mathcal{M} = T_x M \times T_y N \to T_y N$ is the trivial projection. A nonautonomous vector field on $N$ is defined by

$$dY/dt = \tilde{F}_x(t, Y) \ . \tag{17}$$

So given any $Y(0) \in N$, this produces a trajectory $Y(t) \in N$ for $t \geq 0$ (under certain assumptions on $\tilde{F}_x$ and $N$ necessary for the global existence of solutions). This may also be lifted to a motion on $\mathcal{M}$ given by $(x(t), Y(t))$. The definitions of synchronization for vector fields are completely analogous to those for mappings.

We now make a couple of remarks concerning these constructions:

1. A global product structure on $\mathcal{M}$ is not always necessary. For example, one can discuss synchronization of vector fields on the Mobius strip.

2. In all our examples, the product structures chosen for synchronization are trivial, although not necessarily seen from the form in which the dynamical system is most commonly written.

Once the phase space of a dynamical system has been given a product structure, we have demonstrated how to produce a nonautonomous dynamical system. This system must be analyzed in order to determine the synchronization properties of this particular choice of product structure. An alternative viewpoint allows a geometric interpretation, but comes at the cost of increasing the dimension of the dynamical system. We will relate properties of this larger system to our definitions of synchronization.

Suppose that a product structure is chosen so that $\mathcal{M} = M \times N$, and in these coordinates the dynamical system defined by $F$ may be written as in (1), with $x \in M, y \in N$ and

$$\begin{aligned} f &: M \times N \to M \\ g &: M \times N \to N \ , \end{aligned} \tag{18}$$

or

$$\begin{aligned} f &: M \times N \to TM \\ g &: M \times N \to TN \ , \end{aligned} \tag{19}$$

depending on whether we are working with a discrete or continuous dynamical system. We may simultaneously consider the evolution of $Y$, as in (2), by studying the dynamical system $F$ extended to $M \times N \times N$,

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \\ Y' &= g(x,Y) \end{aligned} \tag{20}$$

Let $\mathcal{D} = \{(x,y,Y) | y = Y\}$ which is an invariant submanifold of (20). The synchronization properties of $(\mathcal{M}, F, M, N)$ depend on the asymptotics of (20), and in particular, the attractivity properties of $\mathcal{D}$. The relations between this geometric viewpoint and synchronization are stated in the following theorem.

**Theorem 5** *Every trajectory of (20) is asymptotic to $\mathcal{D}$ if and only if $(\mathcal{M}, F, M, N)$ is synchronizing.*

A geometrical statement which implies absolute synchronization of $(\mathcal{M}, F, M, N)$ is given below. First, let

$$u' = g(\lambda, u) \tag{21}$$

with $u \in N$ and $\lambda \in M$ be the parameterized family of dynamical systems on $N$ where $g$ is as in (18) or (19). We say that the family of systems (21) is *uniformly contracting* if there exists an $\epsilon > 0$ such that for all $\lambda$ and for all trajectories $u_1(t), u_2(t)$,

$$d_N(u_1(t+s), u_2(t+s)) \leq e^{-\epsilon s} d_N(u_1(t), u_2(t)) \ , \tag{22}$$

for all $s, t \geq 0$.

**Theorem 6** *If the family of dynamical systems (21) is uniformly contracting, then $(\mathcal{M}, F, M, N)$ is absolutely synchronizing.*

**Theorem 7** *If $(\mathcal{M}, F, M, N)$ is absolutely synchronizing, then for each $\lambda$, the $\omega$-limit set of (21) consists of either exactly one point, which is a unique globally attracting fixed point, or is empty, which is possible only if $N$ is not compact.*

Recall that the *$\omega$-limit set* of a dynamical system is the set of accumulation points of its forwards orbits.

# 4 Linear systems

Consider the linear dynamical system on $I\!R^{m+n}$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ , \tag{23}$$

with $x \in I\!R^m, y \in I\!R^n$. The nonautonomous system on $I\!R^n$ is constructed as

$$Y' = Cx + DY \ , \tag{24}$$

and

$$(Y - y)' = D(Y - y) \ . \tag{25}$$

So synchronization of this linear system depends only on the eigenvalues of the submatrix $D$.

**Lemma 8** *The coordinate $x$ is synchronizing for (23) if and only if the eigenvalues of $D$ lie in the left half plane, if the dynamical system is a flow, or lie in the unit disk, for a mapping.*

The question we address is when can a linear change of coordinates give us better synchronization properties. Our result is that generically (with respect to the original linear system), a change of coordinates can produce a new linear system which is synchronized using exactly one new coordinate variable. We do this by demonstrating that a coordinate transformation will put generic matrices into a special form which we define here.

**Definition 9** *A matrix in $M_n(I\!R)$ is said to be of* type $\mathcal{S}$ *if it has the following form*

$$\begin{pmatrix} a & b_2 & \cdots & b_n \\ c_2 & d_2 & & 0 \\ \vdots & & \ddots & \\ c_n & 0 & & d_n \end{pmatrix} , \tag{26}$$

*where $\{d_2, \ldots, d_n\}$ are all distinct.*

The basic theorem which says which matrices may be conjugated to this form follows.

**Theorem 10** *If $M \in M_n(I\!R)$ has $n$ distinct eigenvalues then it is $GL_n(I\!R)$-conjugate to a matrix of type $\mathcal{S}$ where the $\{d_i\}$ are arbitrary (but distinct), $a = tr(M) - \sum_{i=2}^n d_i$, and the products $\{b_j c_j\}$ are determined from $M$ and the $\{d_i\}$.*

This theorem follows directly from the work of Hyman Bass which is presented in an Appendix to this paper. We will be able to use this normal form theorem for both linear maps and flows. If we just consider the case of linear maps, then it would have been sufficient for the proof of our main result to use the companion matrix as a normal form. The main result for the synchronization of linear systems is a consequence of Lemma 8 and Theorem 10.

**Theorem 11 (Linear synchronization)** *A linear dynamical system with distinct eigenvalues is synchronizable with one input.*

**Remark**

Any linear matrix in $GL(2,\mathbb{Z})$ is called Anosov if it defines an Anosov map on $\mathbb{T}^2$, *i.e.*, its eigenvalues do not lie on the unit circle. Any Anosov matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is conjugate to $\begin{pmatrix} a+d & -ad+bc \\ 1 & 0 \end{pmatrix}$ in $GL(2,\mathbb{Q})$ and lifts an Anosov map which is synchronizable after a linear change of coordinates if and only if the conjugacy can be chosen in $GL(2,\mathbb{Z})$. When two matrices in $GL(2,\mathbb{Z})$ are conjugate in $GL(2,\mathbb{Q})$ but not in $GL(2,\mathbb{Z})$, then at least one of them covers a torus map which is not synchronizable after a linear change of coordinates. Roy Adler supplied us with one such pair of Anosov maps:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1 \, , \tag{27}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 9 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1 \, . \tag{28}$$

One can check that the first map cannot be synchronized through a linear change of coordinates.

# 5 More examples

### A linear map

Consider the following linear map on $\mathbb{R}^3$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \, . \tag{29}$$

None of the coordinate variables are synchronizing as a consequence of Lemma 8. We also note that the mapping has two eigenvalues outside the unit disk and one inside. We can perform a change of variables to put it into a form of type $\mathcal{S}$. One particular change of coordinates gives

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}' = \frac{1}{4} \begin{pmatrix} 5 & -10 & 1 \\ 20 & 2 & 0 \\ 137 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \, . \tag{30}$$

It is easy to verify that the original and new matrices have the same characteristic polynomial which has distinct roots, and thus they are conjugate. For (30), $u$ is a synchronizing coordinate.

### The standard and Hénon maps

Heagy & Carroll [7] indicate that it is possible to synchronize chaotic Hamiltonian systems by studying the standard map in the chaotic regime. In particular, they conjecture that for the standard map

$$\begin{aligned} I' &= I + \tfrac{k}{2\pi}\sin 2\pi\theta && \bmod 1 \\ \theta' &= \theta + I + \tfrac{k}{2\pi}\sin 2\pi\theta && \bmod 1\ , \end{aligned} \tag{31}$$

the conditional Lyapunov exponent in the angular direction is negative when $0 < k < 2$. Thus, $I$ is a synchronizing coordinate. Conversely, the conditional Lyapunov exponent for coupling through $\theta$ is zero so synchronizing through $\theta$ is not possible.

Consider an orbit $\{(I_n, \theta_n)\}$ of (31). To each such orbit is associated the nonautonomous dynamical system

$$\phi_{n+1} = F_{I_n}(\phi_n) \equiv \phi_n + I_n + \frac{k}{2\pi}\sin 2\pi\phi_n \qquad \bmod 1 \tag{32}$$

on $\mathbb{T}^1 \equiv \mathbb{R}/\mathbb{Z}$. Each iteration can be thought of as a map from $\mathbb{T}^1$ *onto* $\mathbb{T}^1$, so that any (long) composition is also onto $\mathbb{T}^1$, thus cannot be contracting everywhere. The only way synchronization can occur is if there is an asymptotically unique trajectory which is attracting from one side and repelling from the other (which requires a zero Lyapunov exponent). In the particular case when $I_n \equiv 0$, the circle map has two fixed points, hence, the coordinate $I$ cannot be a synchronizing variable in the sense of Section 3. The convergence observed by Heagy and Carroll [7] is likely to be an example of what might be called *measure theoretic synchronization.* In fact, the numerical experiments indicate that the map

$$F_{I_n} \circ F_{I_{n-1}} \circ \cdots \circ F_{I_0} \tag{33}$$

for $n$ large contracts most of the circle to a small arc. The study of the composition of maps, as in (33), where $I_n$ is either random or prescribed, is beyond the scope of this paper.

A simple change of variables

$$\begin{pmatrix} J \\ \phi \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ \theta \end{pmatrix} \bmod 1 \tag{34}$$

for the standard map (31) produces the system

$$\begin{aligned} J' &= \phi && \bmod 1 \\ \phi' &= 2\phi - J + \tfrac{k}{2\pi}\sin 2\pi\phi && \bmod 1\ , \end{aligned} \tag{35}$$

which is trivially absolutely synchronizing for all $k$ with $\phi$ as a synchronizing coordinate.

We also notice that the Hénon map

$$\begin{aligned} x' &= y \\ y' &= 1 - ay^2 - bx \end{aligned} \tag{36}$$

furnishes a trivial example of synchronization with the $y$-coordinate, even in the Hamiltonian case of $b = 1$. Other provably synchronizable Hamiltonian maps are furnished by Anosov diffeomorphisms of the torus like the one described in Section 2 (see also the remark at the end of Section 4).

**The Lorenz equations**

The Lorenz equations (13) are synchronizable with one input. In standard coordinates, both $x$ and $y$ are synchronizing coordinates. Using the $x$ coordinate the system is absolutely synchronizing, while with the $y$ coordinate it is only synchronizing.

When the parameters in the Lorenz equations are selected so that the flow contains a geometrical Lorenz attractor, then almost all trajectories spend an arbitrarily long time in any arbitrarily small neighborhood of the origin, an equilibrium point. One would expect that the consideration of the linearized equations at the origin might help to determine good coordinates for synchronization. The linearization at the origin is given by

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} . \tag{37}$$

In view of Lemma 8, for this linear system $x$ and $y$ are seen to be good synchronizing coordinates, while $z$ is not.

**The Rössler equations**

The Rössler equations [12] are given by

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c) . \end{aligned} \tag{38}$$

Pecora and Carroll [11] note that the neither $x$ nor $z$ are synchronizing coordinates whenever $a > 0$. They study whether $y$ is synchronizing in the case $a = b = 0.2$ and $c \in (3.0, 11.0)$. In particular, they note that for $c = 4.7$, $y$ does not synchronize.

We experimented with the Rössler system by generating random linear coordinates changes and testing for synchronization of the Rössler equations written in the appropriate new coordinates. We easily found new coordinates which work where the common ones fail. For example, with the randomly generated coordinate change,

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} -0.5288133 & -0.7404837 & -0.496776 \\ -0.6433829 & -0.6578195 & -1.916370 \\ -0.07917038 & -1.406477 & -0.8609861 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{39}$$

and $a = b = 0.2, c = 4.7$, our numerical tests indicate that $\tilde{x}$ is a (locally) synchronizing coordinate.

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## Appendix A new matrix normal form

In this appendix, we present the work of Hyman Bass from which Theorem 10 directly follows.

**Theorem A.1** *Let $F$ be a field and*

$$P(T) = T^n - s_1 T^{n-1} + s_2 T^{n-2} + \cdots + (-1)^n s_n \in F[T] \tag{1}$$

*a monic polynomial of degree $n$. Let $d_2, \ldots, d_n \in F$ be distinct. Then there exists*

$$\alpha = \begin{bmatrix} a & b_2 & \cdots & b_n \\ c_2 & d_2 & & 0 \\ \vdots & & \ddots & \\ c_n & 0 & & d_n \end{bmatrix} \in M_n(F) \tag{2}$$

*with characteristic polynomial*

$$P_\alpha(T)(\equiv \ Det(T \cdot I - \alpha)) = P(T) \ . \tag{3}$$

*Further, $a = s_1 - (d_2 + \cdots + d_n)$ , and only $\beta_j = b_j c_j \ \ (j = 2, \ldots, n)$ are determined by (3).*

**Corollary A.2** *If $P$ in (1) has distinct roots $\lambda_1, \ldots, \lambda_n \in F$ then $\alpha$ in (2) is $GL_n(F)$-conjugate to* $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

**Proposition A.3** *Let $F \subset E$ be infinite fields, and let $\alpha, \beta \in M_n(F)$. If $\alpha$ and $\beta$ are $GL_n(E)$-conjugate, then they are $GL_n(F)$-conjugate.*

These three results are sufficient to prove Theorem 10. We next present the proof of the proposition, then we state a few preliminary results, and finally we present the proof of Theorem A.1.

**Proof (of Proposition A.3)** Let

$$L = \{g \in M_n(F) | g\alpha - \beta g = 0\} \ , \tag{4}$$

a linear subspace of $M_n(F)$. Let

$$D = \mathrm{Det} : M_n(F) \to F \ , \tag{5}$$

a polynomial function. Extending the base field from $F$ to $E$ we obtain

$$L_E = \{g \in M_n(E) | g\alpha - \beta g = 0\} \ . \tag{6}$$

By assumption, there is a $g \in L_E$ such that $D(g) \neq 0$. Thus, $D$ does not vanish on $L_E$. Since $D|_{L_E}$ is a polynomial function on $L_E$, and since, $F$ being infinite, $L$ is Zariski dense in $L_E$, we have $D|_L \neq 0$. So $D(g) \neq 0$ for some $g \in L$, and for this $g$ we have $\beta = g\alpha g^{-1}$. □

**Lemma A.4** *For any* $\alpha = \begin{bmatrix} a & b_2 & \cdots & b_n \\ c_2 & d_2 & & 0 \\ \vdots & & \ddots & \\ c_n & 0 & & d_n \end{bmatrix}$,

$$Det(\alpha) = ad_2 \cdots d_n - \sum_{j=2}^{n} \beta_j d_2 \cdots d_{j-1} d_{j+1} \cdots d_n \ , \tag{7}$$

*with* $\beta_j = b_j c_j$.

**Proof** Expand along the first row:

$$\mathrm{Det}(\alpha) = ad_2 \cdots d_n + \sum_{j=2}^{n} (-1)^{j+1} b_j \mathrm{Det} \begin{bmatrix} c_2 & d_2 & & & & & \\ \vdots & 0 & \ddots & & & & \\ \vdots & & \ddots & d_{j-1} & & & \\ c_j & & & 0 & & & \\ c_{j+1} & & & & d_{j+1} & & 0 \\ \vdots & & & & & \ddots & \\ c_n & & & & 0 & & d_n \end{bmatrix} . \tag{8}$$

The signature of the $(j-1)$ cycle $(2, 3, \ldots, j)$ is $(-1)^j$, so

$$\mathrm{Det}\begin{bmatrix} c_2 & d_2 & & 0 \\ \vdots & 0 & \ddots & \\ \vdots & & \ddots & d_{j-1} \\ c_j & 0 & & 0 \end{bmatrix} = (-1)^j c_j d_2 \cdots d_{j-1} \ , \tag{9}$$

whence the lemma. $\square$

**Corollary A.5** *$P_\alpha(T)(\equiv Det(T \cdot I - \alpha)) = (T-a)D(T) - \sum_{j=2}^{n} \beta_j D_j(T)$, where*

$$D(T) = (T-d_2)\cdots(T-d_n) = (T-d_j)D_j(T) \ \ (j = 2, \ldots, n) \ . \tag{10}$$

**Proof** Apply Lemma A.4 to

$$T \cdot I - \alpha = \begin{bmatrix} T-a & -b_2 & \cdots & -b_n \\ -c_2 & T-d_2 & & 0 \\ \vdots & & \ddots & \\ -c_n & 0 & & T-d_n \end{bmatrix} \ . \tag{11}$$

$\square$

**Corollary A.6** *$P_\alpha(d_j) = -\beta_j D_j(d_j) \ \ (j = 2, \ldots, n)$.*

**Proof (of Theorem A.1)** Given $d_2, \ldots, d_n$ distinct and $P(T) = T^n - s_1 T^{n-1} + \cdots \in F[T]$ monic of degree $n$, construct

$$\alpha = \begin{bmatrix} a & b_2 & \cdots & b_n \\ c_2 & d_2 & & 0 \\ \vdots & & \ddots & \\ c_n & 0 & & d_n \end{bmatrix} \tag{12}$$

so that

$$a = s_1 - (d_2 + \cdots d_n) \tag{13}$$

and

$$P(d_j) = -\beta_j D_j(d_j) \ \ (j = 2, \ldots, n) \ , \tag{14}$$

where $\beta_j = b_j c_j$ and

$$D(T) = (T-d_2)\cdots(T-d_n) = (T-d_j)D_j(T) \ \ (j = 2, \ldots, n) \ . \tag{15}$$

Note that since $d_2, \ldots, d_n$ are distinct, $D_j(d_j) \neq 0$, so $\beta_j = -P(d_j)/D_j(d_j)$ is defined by (14).

From Corollary A.6 and (14) we have

$$P(d_j) = -\beta_j D_j(d_j) = P_\alpha(d_j) \quad (j = 2, \ldots, n) \ , \tag{16}$$

so $d_2, \ldots, d_n$ are $n-1$ distinct roots of

$$Q(T) = P(T) - P_\alpha(T) \ . \tag{17}$$

Since $P$ and $P_\alpha$ are monic of degree $n$ and the coefficient $-s_1 = -(a + d_2 + \cdots + d_n) = -\mathrm{Tr}(\alpha)$ of $T^{n-1}$ is the same for $P$ and $P_\alpha$, we have $\deg(Q) \leq n-2$. Hence $Q = 0$, and $P_\alpha = P$. □

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