Resynchronizing dynamical systems

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Abstract

Resynchronizing dynamical systems are important for certain chaotic signal masking methods. We demonstrate the dependence of the resynchronizing property of linear dynamical systems on choices of coordinate systems. Using these insights, we demonstrate how a nonlinear system not previously known to be synchronizable can be used for chaotic signal masking.

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Resynchronizing dynamical systems are important for the chaotic signal masking method proposed by Cuomo, Oppenheim and Strogatz [1] and improved upon by Wu and Chua [2]. This method relies on a notion of synchronization in dynamical systems introduced by Pecora and Carroll [3] and previously called by the current authors "master-slave synchronization" [4]. For two additional and completely different types of signal masking with chaotic dynamical signals see [5,6]. Essentially, a resynchronizing dynamical system is one in which there are two synchronizing coordinates. This means that a driving signal may be used to synchronize a subsystem and then a signal from the subsystem can synchronize the remaining coordinates, thus reproducing all the driving signals. Resynchronizing systems have also been used (but not by this name) by Pecora and Carroll [7] to construct a cascaded synchronizing chaotic system. In follow up work [8] they showed how cascaded resycnhronizing systems can be used to track parameter variations in a chaotic analog of a phase-locked loop. Parlitz, *et. al.*, [9] suggested an application of resynchronizing dynamical systems to the transmission of digital signals, where binary 1's and 0's are encoded as synchronizing and desynchronizing states. Here we present an analysis of linear resynchronizable systems in the style of our work on synchronizing systems [4] and then use our insight from the linear analysis to show that a specific nonlinear system not known to be resynchronizable may in fact be used for chaotic signal masking.

We now recall Pecora and Carroll's notion of synchronization and demonstrate how these ideas are used by Wu and Chua for signal masking. Let

$$\begin{aligned} x' &= f(x, y)\,, \\ y' &= g(x, y)\,, \end{aligned} \tag{1}$$

be evolution equations of some smooth dynamical system $\mathcal{S}$, where x and y are respectively m and n-dimensional vectors, and where the prime stands either for differentiation with respect to time (in the case of a vector field) or for the next value (in the case of a map). With t standing for either continuous or discrete time (according to whether we deal with a vector field or a map), and $X(t)$ standing for some m-dimensional smooth function of time, we can construct a time-dependent n-dimensional evolution equation

$$Y' = g(X(t), Y)\,. \tag{2}$$

Let Y_α be an arbitrary initial condition and $Y_\alpha(t)$ be the corresponding solution for Eq. (2). If for every pair (Y_α, Y_β), $\| Y_\alpha(t) - Y_\beta(t) \| \rightarrow 0$ as $t \rightarrow \infty$ when $(X(t), \cdot)$ is a solution of Eq. (1), we say that the evolution equations are *synchronizing* (with *synchronizing variable* x) [3]. If for every pair (Y_α, Y_β), $\| Y_\alpha(t) - Y_\beta(t) \| \rightarrow 0$ as $t \rightarrow \infty$ for any $X(t)$, we say that the evolution equations are *absolutely synchronizing* (with *absolutely synchronizing variable* x) [4]. We say that the dynamical system $\mathcal{S}$ is (*absolutely*) *synchronizable* if its evolution equations in some coordinate system are (absolutely) synchronizing. In some cases, like for the dynamical system defined by the Rössler equations that we use below as an example, one might need to restrict the dynamics to some subset of the phase space in order to get a synchronizable dynamical system.

Inspired by the work of Cuomo and Oppenheim [10] (see also [2] and references therein), we say that the equations Eq. (1) are *resynchronizing* if both x and y are synchronizing variables. In other words, the y subsystem, as given in Eq. (2) is synchronizing when driven with $x(t)$ and the corresponding x subsystem is synchronizing when driven with $y(t)$. The value of a resynchronizing system is that it can be synchronized with more than one coordinate and all driving signals can be recovered. Similarly, we call a system *absolutely resynchronizing* if x and y are both absolutely synchronizing coordinates. An example of a resynchronizing dynamical system is the Lorenz equations [3,4,10]. This follows from the fact that the Lorenz equations have two different synchronizing coordinates: the y, z subsystem is synchronizing when driven by the x coordinate and the x, z subsystem is synchronizing when driven by the y coordinate.

In practice we would like all our synchronization phenomenon to be "stable." This may be made formal in the sense of Artstein [11,12]. Essentially, suppose that $||X_1(t) - X(t)|| \rightarrow 0$ as $t \rightarrow \infty$ then we need from Eq. (2) that $Y_1' = g(X_1(t), Y_1)$ admits $Y' = g(X(t), Y)$ as a limiting equation. From this and standard regularity assumptions, absolute synchronziability of the

x-coordinate in Eq. (1) implies that $||Y_1(t) - Y(t)|| \to 0$ as $t \to \infty$.

As an example of the difference between synchronizability and resynchronizability, let us mention that while an open dense subset of the linear maps or flows in $\mathbb{R}^n$, $n > 1$, are synchronizable by a linear change of variables [4], the same methods from [4] allows one to get the following results:

Theorem 1 *An open dense subset of the volume contracting linear flows in* $\mathbb{R}^n$*,* $n > 1$*, are resynchronizing after a linear change of variables. A linear flow in* $\mathbb{R}^n$*,* $n > 1$ *which is not volume contracting cannot be transformed to a resynchronizing one by a linear change of variables.*

Theorem 2 *An open dense subset of the linear maps in* $\mathbb{R}^n$*,* $n > 1$*, whose trace has absolute value strictly smaller than* n *are resynchronizing after a linear change of variables. A linear map in* $\mathbb{R}^n$*,* $n > 1$*, whose trace* T *satisfies* $|T| \geq n$ *cannot be transformed to a resynchronizing one by a linear change of variables.*

These theorems indicate that for most linear systems a change of coordinates can produce a (re)synchronizing one even if the system written in the original coordinates was not (re)synchronzing. One could attempt to apply the same strategy to a nonlinear system. In [4] this was demonstrated in this case of synchronization and later in this paper we shall demonstrate it for resynchronization. Carroll, *et. al.*, [13] use transformations of the driving signal, rather than coordinate transformations, in order to achieve synchronization in systems which are not naturally synchronizable.

Several methods for transmitting information using a chaotic signal as a broadband carrier and synchronizing systems have been suggested and implemented. See, for example, the work of Koacarev, *et. al.*, [14] and Kocarev and Parlitz [15] and the references therein. We shall recall Wu and Chua's method [2] of chaotic signal masking, which improved upon the method presented by Cuomo, Oppenheim and Strogatz [1]. Suppose that you have two identical dynamical systems, one which shall be called the "transmitter" and the other the "receiver." With the transmitter, one has a message $m(t)$ and a smooth coding function $C : \mathbb{R}^2 \to \mathbb{R}$ which combines the message with a carrier signal $x(t)$. This produces a signal $s(t) = C(x(t), m(t))$. The function C must be invertible with smooth inverse so that if one knows $x(t)$ and $s(t)$ then $m(t)$ is recoverable. In other words there exists a smooth decoding function $D : \mathbb{R}^2 \to \mathbb{R}$ so that $m(t) = D(x(t), C(x(t), m(t)))$. Consider the folowing setups as the transmitter:

$$
\begin{aligned}
x' &= f(x, y) \\
y' &= g(s(t), y) \\
s(t) &= C(x, m(t)) \ ,
\end{aligned}
$$

and as the receiver:

$$
\begin{aligned}
X' &= f(X, Y) \\
Y' &= g(s(t), Y) \\
\hat{m}(t) &= D(X, s(t)) \ .
\end{aligned}
$$

If x is an absolutely synchronizing coordinate then we see that $||Y(t) - y(t)|| \to 0$ as $t \to \infty$. If y is a stably absolutely synchronizing coordinate then also $||X(t) - x(t)|| \to 0$ as $t \to \infty$. The "resynchronizing" property has allowed us to reconstruct both the original x signal and the original y signal. Consequently, by continuity of the decoding function D, we find that $||\hat{m}(t) - m(t)|| \to 0$ as $t \to \infty$. We see that any stably absolutely resynchronizing system can be used for this method of signal masking. The systems of interest are those for which the signal $x(t)$ is chaotic so that mixing $x(t)$ and $m(t)$ in the coding function C will produce a signal $s(t)$ which appears chaotic.

The requirement we made that y be an absolutely synchronizing variable is stronger than needed, and in fact does not hold in the example presented below. Since, on the other hand, it is not sufficient to assume that y is a synchronizing variable, we chose to present the theory with a sufficient, stronger property already introduced in [4]; a precise discussion of such matters is beyond the scope of this short report. For the remainder of this report we shall loosely use the term resynchronizing to mean that the system can be used for signal masking in the previously described way. For some systems, as in the example below, we could not prove the resynchronizing property, but merely observe it experimentally.

The method of chaotic signal masking described here and in the related papers has been seriously attacked from the cryptographic point of view in [16–18]. On the other hand, as described in [19,20], there are applications other than secrecy for which the possibility of masking a signal by chaotic data, and then recovering it, is valuable (*e.g.*, improved resistance to interference and greater channel sharing).

The value for signal masking of the simple observations reported here depends primarily on one's ability to find resynchronizing coordinate systems. We wish to emphasize the results from [4] that synchronizability of a dynamical system is tied very closely to the choice of coordinates in which one chooses to write the system. A characterization of the relative abundance of resynchronizable dynamical systems among the dissipative systems in $\mathbb{R}^n$ has so far eluded our investigations. However, we notice that we can find resynchronizing coordinates (in some reasonable subset of their phase space) for some systems which previously were not even recognized as being synchronizable [21], as in the following example.

We consider the Rössler equations

$$\begin{aligned}\dot{x} &= -(y+z)\,,\\ \dot{y} &= x + ay\,,\\ \dot{z} &= b + z(x-c)\,,\end{aligned}$$

with $a = b = 0.2$ and $c = 4.7$. It is easy to show that the x and z coordinates are not synchronizing. In [21], it was demonstrated that the y coordinate is sometimes synchronizing, depending on the choice of parameter values. For these particular parameter values, no coordinate is synchronizing. However, it was shown in [4] that this dynamical system written down in a suitable set of coordinates is synchronizing. Furthermore, with a random search we easily find the change of variables

$$u = 0.385x + 0.536y - 0.940z\,,$$

$$v = 0.908x + 0.233y + 0.641z\,,$$

$$w = -0.518x - 0.974y - 0.554z\,,$$

such that the pair $(u, (v, w))$ is resynchronizing. Fig. 1 illustrates the results of applying this dynamical system and the signal masking scheme to a simple sinusoidal message. For this example the coding function C is given by $C(u, m) = u + m$ and the decoding function is then $D(U, s) = s - U$. We observe that after a small transient, the message is decoded accurately. In practice, we do not observe asymptotic convergence of the decoded message to the real message. However, the error is several orders of magnitude less than the magnitude of the message which was in turn two orders of magnitude less than the magnitude of the carrying signal. The size of the message and how well it can be reconstructed depends of course on the synchronization properties of the dynamical system. For an alternate method of chaotic signal masking with excellent noise to signal ratios that does not depend on synchronization, refer to [6].

It is easily seen that resynchronizing dynamical systems can be used in control applications. Suppose that the signal passed is simply the x-coordinate, *i.e.* $s(t) = x(t)$. Then the receiving system will converge to the trajectory of the transmitting system. Thus given a resynchronizing dynamical system and $x_0(t)$ which is the x-coordinate of a trajectory to which we wish to target, we can force the system in the following way:

$$\begin{aligned} X' &= f(X, Y) \\ Y' &= g(X, Y) + F(t) \end{aligned} \tag{3}$$

where

$$F(t) = g(x_0(t), Y) - g(X, Y)\,.$$

The forcing which has to be imposed to stabilize a given orbit is small if and only if the actual trajectory is close to the desired one and in phase with it. If the desired orbit has a dense stable manifold and the simplest use of Eq. (3), where $x_0(t)$ specifies the desired orbit, requires a forcing considered as being too big, one can impose appropriate trajectories on this manifold to target the desired orbit. There is then a tradeoff between the length of the branch of stable manifold one wants to use and the maximal strength of forcing one is willing to impose. In particular, for some applications where one wants to stabilize a periodic orbit whose phase is irrelevant, one can use a feedback system allowing one to target the desired orbit more efficiently.

Synchronization has been used before in control applications in a less restrictive manner than presented here. Rulkov, Tsimring and Abarbanel [22] use a form of linear feedback which accomplishes control in synchronizing systems in an especially elegant way. Nevertheless, depending on which state variables are physically accessible for control, the method proposed here may be useful.

Similar cancellation methods for control of nonlinear systems that do not take advantage of synchronization phenomenon have been studied. Hübler and Lüscher [23] proposed controlling nonlinear dynamical systems by adding a term which cancels all components of the vector field and replaces it with some desired behavior. Jackson [24] studies this control method for regions of the space where there is convergence of trajectories and considers its application to multiple attractor systems. The approach of cancelling one term and replacing it with a desired one is common in the control literature [25].

We have applied our control strategy to the Rössler equations in resynchronizing coordinates. Fig. 2 illustrates how one reaches the close neighborhood of a periodic orbit by imposing the $u_0(t)$ of this orbit (up to the error in the approximation), while in Fig. 3, starting from the same initial condition on v and w, one imposes $u_0(t+\tau)$ were τ is adjusted to diminish the initial forcing. It is interesting to compare the time series of the forcing needed in both cases. We note that in most practical cases the forcing will become small but will not converge to zero because in practice one often only inputs an approximation to a true orbit.

A precise and/or systematic comparison between the methods proposed in this paper and the so-called OGY method [26] is beyond the scope of this report. We just emphasize here that the resynchronization-based method that we propose does not necessarily require one to compute first return maps, except possibly for the purpose of accelerated targeting as we mentioned earlier.

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FIGURES

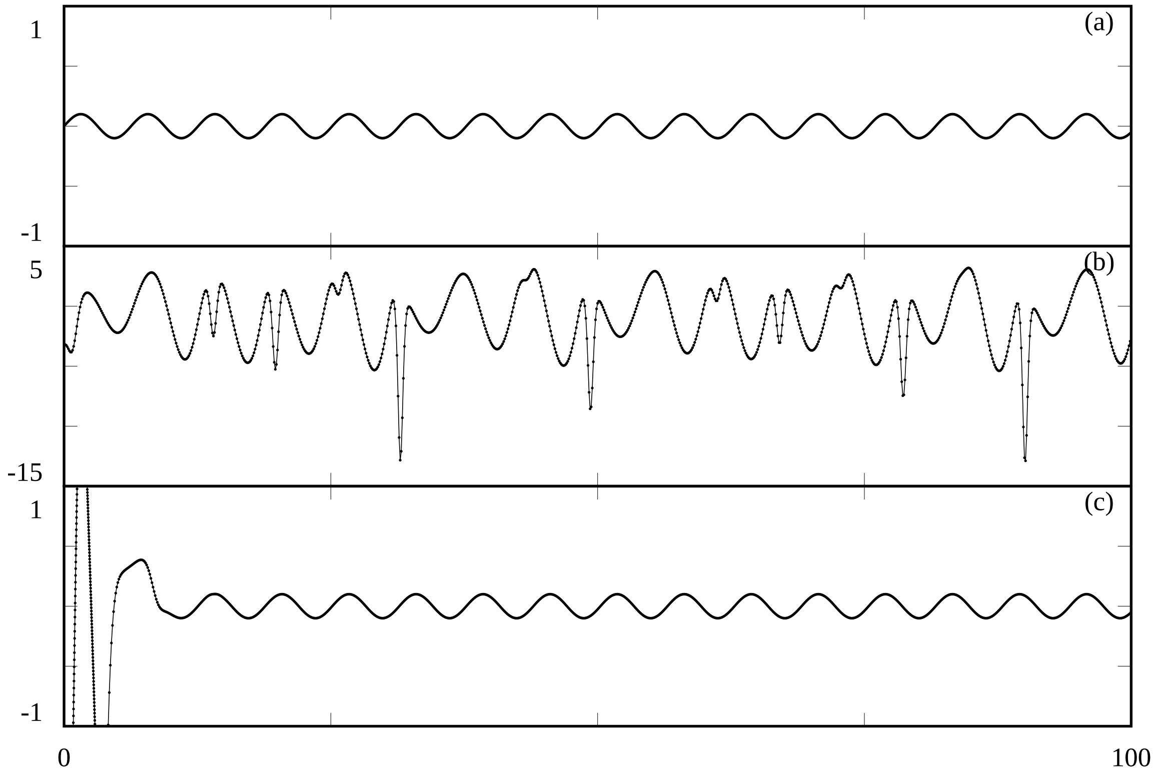

FIG. 1. (a) Time series of message; (b) Time series of chaotic signal containing the message; (c) Time series of the decoded message from (b).

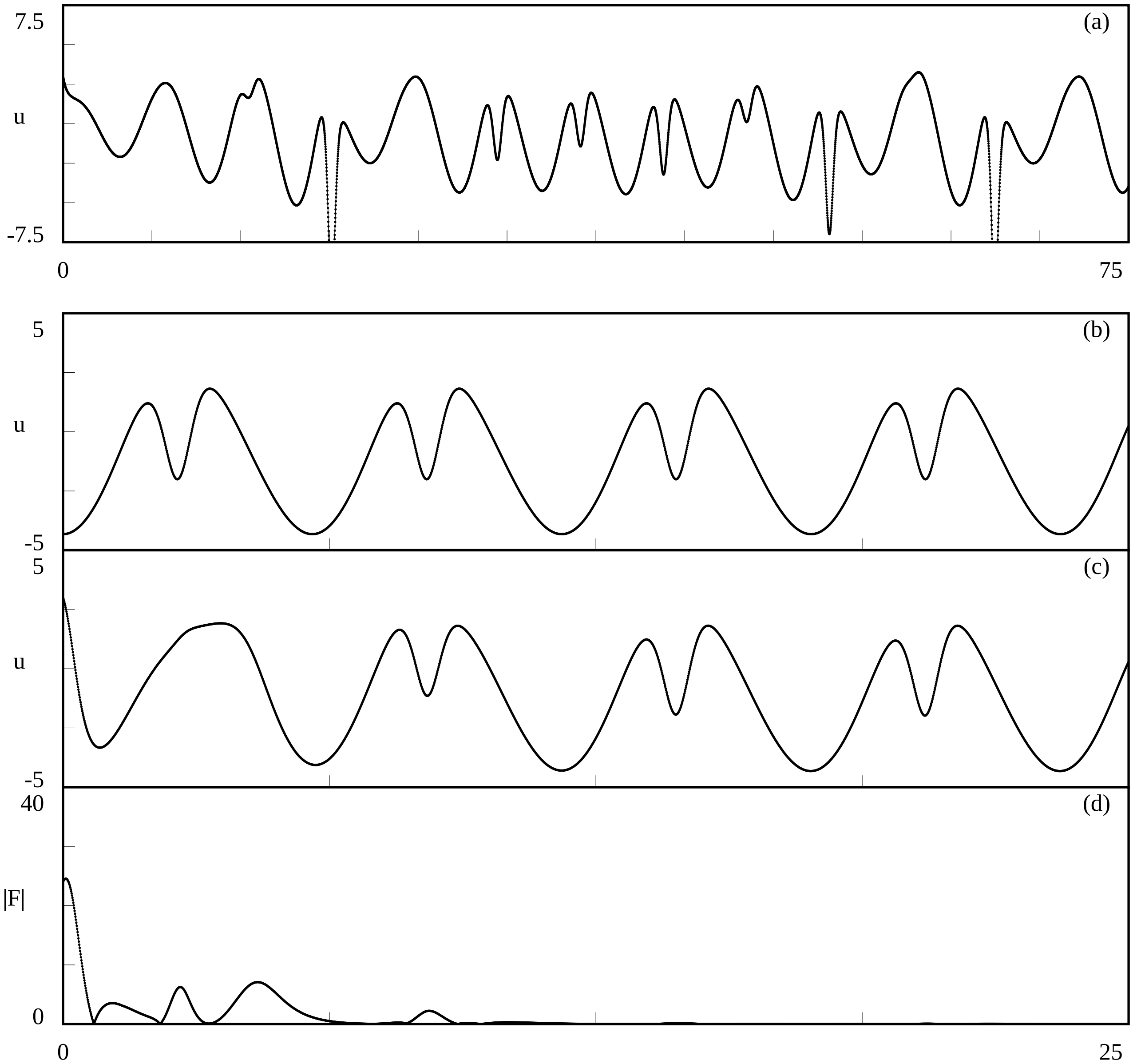

FIG. 2. (a) Time series of one coordinate of a typical trajectory of the Rössler system; (b) Time series of one coordinate of a periodic orbit which we will stabilize; (c) Time series of one coordinate from the same initial condition as (a) with the control on; (d) Time series of the magnitude of the forcing corresponding to (c).

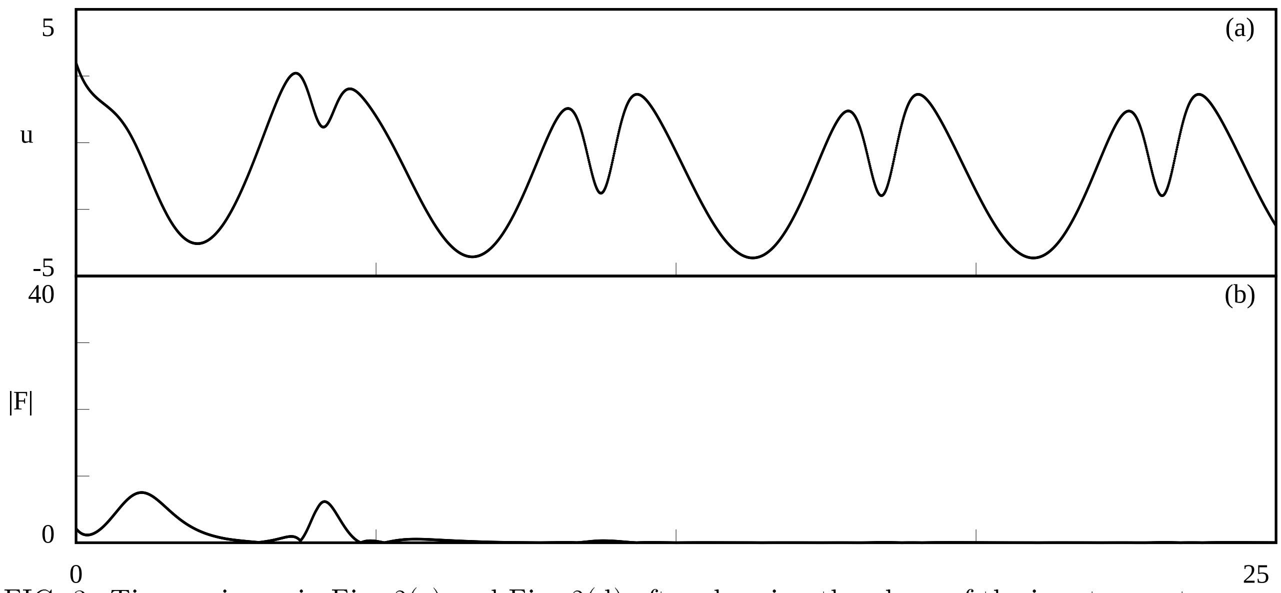

FIG. 3. Time series as in Fig. 2(c) and Fig. 2(d) after choosing the phase of the input so as to diminish the forcing.