AN EQUIVARIANT, INCLINATION-FLIP, HETEROCLINIC BIFURCATION

PATRICK A. WORFOLK
THE GEOMETRY CENTER
UNIVERSITY OF MINNESOTA
1300 S. SECOND ST., MINNEAPOLIS, MN 55454

Abstract. We examine a heteroclinic bifurcation occurring in families of equivariant vector fields. Within these families, the flows contain structurally stable heteroclinic cycles. The flow can twist around the cycle to produce what is the equivalent of an "inclination-flip" homoclinic orbit. An unfolding of this bifurcation in a generic one-parameter equivariant family shows that Smale horseshoes are embedded in the flow for an open set of parameter values.

Short title: Equivariant heteroclinic bifurcation

1. Introduction

In this paper we study an equivariant heteroclinic bifurcation which produces chaotic dynamics. When the quotient of the phase space with respect to the group action is taken, the heteroclinic cycles we consider are reduced to homoclinic orbits, so we may conduct our analysis within the mathematical framework of homoclinic bifurcation. This heteroclinic bifurcation has two special features. The first is that the heteroclinic orbits are not broken because they are artifacts of the symmetry. The second is that the flow in a neighborhood of the heteroclinic orbits respects a rotational symmetry. Consequently, we are not studying the equivalent of a generic codimension two homoclinic bifurcation, but are looking at a generic, codimension one, equivariant heteroclinic bifurcation. Equivariant homoclinic bifurcations have been studied before and, for example, one type occurs in the Lorenz equations [10] (where the symmetry is a simple rotation through π). However, this particular bifurcation is new and predicts chaotic behavior.

We will be studying bifurcations in one-parameter families of vector fields which are generic within a class of vector fields with symmetry. Each element of the family has the same symmetry as defined by the following:

Definition 1.1. *Let G be a finite subgroup of $O(n)$ acting on $\mathbb{R}^n$. A vector field $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, is G-equivariant or symmetric if*

$$f(gx) = gf(x) \text{ for all } g \in G \ .$$

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For a fixed group G, we wish to study generic bifurcation, both local and global, within the class of G-equivariant vector fields. In other words, we are interested in the bifurcations which persist under sufficiently small G-equivariant perturbations.

Consider the action of a group $\tilde{G}$ on $\mathbb{R}^3$ generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and studied by Guckenheimer and Holmes [9]. A few short calculations show that the degree three $\tilde{G}$-equivariant normal form may be written as:

$$\begin{aligned} \dot{x}_1 &= (a_0 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2) x_1 \\ \dot{x}_2 &= (a_0 + a_1 x_2^2 + a_2 x_3^2 + a_3 x_1^2) x_2 \\ \dot{x}_3 &= (a_0 + a_1 x_3^2 + a_2 x_1^2 + a_3 x_2^2) x_3 \ . \end{aligned} \tag{1}$$

Guckenheimer and Holmes observed that this system of equations has structurally stable heteroclinic cycles for open regions of the parameter space.

Theorem 1.2. [9] *If either $a_3 < a_1 < a_2 < 0$ or $a_2 < a_1 < a_3 < 0$ and $2a_1 > a_2 + a_3$ then Equations 1 have attracting heteroclinic cycles for $a_0 > 0$ and a globally attracting equilibrium at the origin for $a_0 \leq 0$.*

This theorem essentially follows from realizing that the coordinate planes are invariant and examining the flow in them. The consequence is that within families of $\tilde{G}$-equivariant vector fields there are generic one-parameter bifurcations from an attracting equilibrium point to structurally stable attracting limit cycles.

Now consider the action of $\bar{G}$ on $\mathbb{R}^4$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and studied by Field and Swift [6]. The cubic $\bar{G}$-equivariant normal form is given by

$$\begin{aligned} \dot{x}_1 &= (a_0 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2) x_1 \\ \dot{x}_2 &= (a_0 + a_1 x_2^2 + a_2 x_3^2 + a_3 x_4^2 + a_4 x_1^2) x_2 \\ \dot{x}_3 &= (a_0 + a_1 x_3^2 + a_2 x_4^2 + a_3 x_1^2 + a_4 x_2^2) x_3 \\ \dot{x}_4 &= (a_0 + a_1 x_4^2 + a_2 x_1^2 + a_3 x_2^2 + a_4 x_3^2) x_4 \ . \end{aligned} \tag{2}$$

If $a_1 = a_2 = a_3 = a_4 = -1$ and if $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, then $\dot{r} = a_0 r - r^3$. So for $a_0 \leq 0$ the origin is a globally attracting equilibrium point and for $a_0 > 0$ there is an invariant 3-sphere of radius $\sqrt{a_0}$. The invariant sphere is normally hyperbolic and all trajectories which are not identically zero are asymptotic to it. If $a_1, \ldots, a_4$ are close to -1, then the sphere will persist as an invariant topological 3-sphere in the post bifurcation flow. This is also true for Equations 1 and a stronger result is given in the invariant sphere theorem by Field and Swift.

Theorem 1.3. *(Invariant Sphere Theorem)* [7] *Let $\dot{x} = a_0 x + Q(x)$ be a G-equivariant vector field where $Q(x)$ is a homogeneous cubic polynomial equivariant. If $(Q(x), x) < 0$ for all $x \neq 0$ ($Q(x)$ is contracting), then for every $a_0 > 0$ there exists a unique $(n-1)$-dimensional sphere $\mathcal{S}(a_0) \subset \mathbb{R}^n - \{0\}$ which is invariant under the flow. This topological sphere is globally attracting in the sense that every trajectory which is not identically zero is forward asymptotic to $\mathcal{S}(a_0)$. Furthermore, the flow on $\mathcal{S}(a_0)$ is topologically conjugate to the flow of $u' = Q(u) - (Q(u), u)u$, where $u \in S^{n-1} \equiv \{u \in \mathbb{R}^n : |u| = 1\}$.*

This theorem allows us to study the dynamics on the invariant sphere. Those features which are robust will be generic properties of the equivariant bifurcation. In other words, these features will not be removed by small perturbations or the addition of higher order terms to the normal form. Varying the parameters a_i in Equation 2, reveals a wide range of behavior on the sphere including limit cycles and heteroclinic cycles with a variety of different stability properties. This analysis by Field and Swift proved the existence of a stationary bifurcation to limit cycles.

This brings us to the symmetry group which is the focus of this paper. Consider the group $G \subset \bar{G}$ comprised of the elements $\bar{G}$ which are orientation preserving. A new term must be added to the cubic equivariant normal form given in Equation 2 which results in

$$\begin{aligned}
\dot{x}_1 &= (a_0 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2)x_1 + a_5 x_2 x_3 x_4 \\
\dot{x}_2 &= (a_0 + a_1 x_2^2 + a_2 x_3^2 + a_3 x_4^2 + a_4 x_1^2)x_2 - a_5 x_1 x_3 x_4 \\
\dot{x}_3 &= (a_0 + a_1 x_3^2 + a_2 x_4^2 + a_3 x_1^2 + a_4 x_2^2)x_3 + a_5 x_1 x_2 x_4 \\
\dot{x}_4 &= (a_0 + a_1 x_4^2 + a_2 x_1^2 + a_3 x_2^2 + a_4 x_3^2)x_4 - a_5 x_1 x_2 x_3 \,.
\end{aligned} \tag{3}$$

The potential for complex dynamics of this normal form was communicated to us by Mike Field (also see [8, Remark 15.8]). These dynamics appear to be very rich and were examined by Guckenheimer and Worfolk in [11]. Once again, for $a_1, \ldots, a_4$ close to -1, $a_0 > 0$, and all values of a_5, there is an invariant, attracting, topological sphere in the flow on which one can study the dynamics. As in the earlier examples, there are structurally stable heteroclinic cycles and limit cycles. The work [11] includes a numerical study which reveals unmistakable signs of chaotic behavior: period doubling cascades and Šilnikov homoclinic orbits. This leads to the conclusion that the flow on the invariant sphere can contain hyperbolic chaotic sets (embedded Smale horseshoes) and nonhyperbolic chaotic attractors. Due to the persistence of hyperbolic sets, we can expect to bifurcate directly from a trivial equilibrium to a flow containing chaotic invariant sets of small amplitude.

In this paper we prove that there exist generic codimension one heteroclinic bifurcations in G-equivariant vector fields which create hyperbolic chaotic invariant sets (embedded Smale horseshoes) in the flow. We examine G-equivariant vector fields which contain structurally stable heteroclinic cycles between equilibria in the same G-orbit. We will call these cycles *homoclinic cycles* since they are homoclinic orbits in the quotient space $\mathbb{R}^4/G$. Unlike standard homoclinic bifurcation, the cycles do not break under small parameter variations. Instead, we are interested in small perturbations of a vector field with a degenerate twist in the flow along a homoclinic cycle. We will give conditions for a one-parameter family of vector fields to contain this bifurcation. The second section of this paper presents the geometric aspects of the bifurcation which is a discussion of how the horseshoes are created and we state the main theorem. In the third section we analyze a Poincaré return map in order to prove the main theorem. In the fourth section, we discuss the analytic problems associated with proving that a specific family of vector fields contains the bifurcation. In particular, we can prove that our cubic normal form satisfies all but one of the geometric conditions presented in the second section. We believe that it satisfies all the conditions, but have been unable to prove one of the nondegeneracy conditions.

The creation of horseshoes in a codimension two homoclinic bifurcation with a twist degeneracy has been studied by Deng [3] and Homburg *et. al.* [14]. The strategy used to study homoclinic cycle bifurcations was introduced by Šilnikov. We take a codimension one cross section to the homoclinic cycle and study a return map on the section. This style of analysis leads to basic results on the asymptotic stability of the heteroclinic cycles. The stability is determined by the flow near the equilibrium points and is therefore a local analysis. The existence of more complicated behavior generally depends on the flow along the homoclinic cycle and is a global and harder analysis.

This is a standard method for constructing a geometric model, but analytically proving the existence of such a global model in a particular family of equations is still a problem fraught with technical difficulties. Our presentation is similar to the work of Rychlik [17] and Robinson [15, 16]. In their work they prove that families of equations closely related to the Lorenz equations contain a homoclinic bifurcation to a flow containing geometric Lorenz attractors.

2. The geometric model

In this section we present the geometric model for an equivariant homoclinic bifurcation. We give a set of conditions which if a family of G-equivariant vector fields satisfies will guarantee the existence of horseshoes in the flow for an open set of parameters. Let $C^r_G(\mathbb{R}^4, \mathbb{R}^4)$ denote the set of all G-equivariant vector fields of class C^r. For our geometric analysis we need only that $r \geq 3$, but we are specifically interested in analytic vector fields since we wish to apply this analysis to our family of cubic normal forms. To avoid the overuse of subscripts, we will write the cubic G-equivariant vector field after rescaling time (to remove the parameter a_0) as

$$\dot{x} = x + Q(x, a, b, c, d, e) \ ,$$

where

$$Q(x,a,b,c,d,e) = \begin{pmatrix} ax_1r^2 + bx_1x_2^2 + cx_1x_3^2 + dx_1x_4^2 + ex_2x_3x_4 \\ ax_2r^2 + bx_2x_3^2 + cx_2x_4^2 + dx_2x_1^2 - ex_1x_3x_4 \\ ax_3r^2 + bx_3x_4^2 + cx_3x_1^2 + dx_3x_2^2 + ex_1x_2x_4 \\ ax_4r^2 + bx_4x_1^2 + cx_4x_2^2 + dx_4x_3^2 - ex_1x_2x_3 \end{pmatrix}$$

$$r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \ .$$

We now recall some of the dynamics of the cubic normal form in order to provide a framework within which to introduce the geometric features required for the heteroclinic bifurcation.

We shall want to apply the invariant sphere theorem to the normal form so shall consider only values of the parameters such that $Q(x) = Q(x,a,b,c,d,e)$ is contracting. The conditions under which $Q(x)$ is contracting are given in

Lemma 2.1. *([6], Lemma 3.1) $Q(x)$ is contracting (i.e., $(Q(x),x) < 0$ for all $x \neq 0$) if and only if*

$$a < 0,\ 2a + c < 0,\ 4a + b + d < 0,\ 4a + b + c + d < 0 \ .$$

When $a < 0$, the variable x may be rescaled so that $a = -1$ and we shall henceforth assume this. If $b = c = d = 0$, then the attracting invariant sphere is in fact the unit sphere and it is normally hyperbolic. So for b, c, d sufficiently small the topological invariant sphere will exist and be attracting and will also be arbitrarily smooth. The smoothness of the sphere is determined by the eigenvalues at the equilibrium points and we will in general require only that it is C^3. An additional consequence of the proof of the invariant sphere theorem is that the sphere $\mathcal{S}$ may be represented as a graph over the unit sphere (*i.e.*, $\mathcal{S} = \rho(u)u$ where $u \in \{x \in \mathbb{R}^4 : |x| = 1\}$ and $\rho(u) \in \mathbb{R}^+$). Since the sphere $\mathcal{S}$ is invariant and attracts all points except the origin, we shall restrict our attention to the dynamics on the sphere. This reduces the dimension of the phase space by one and allows for easier visualization and understanding of the flow.

Any G-equivariant vector field will contain subspaces which are forced to be invariant because of the symmetry. These include the coordinate axes and the coordinate planes. If subspaces map to each other by elements of the symmetry group, then the dynamics within the subspaces will too. Thus only one representative subspace needs to be considered. This allows us to easily prove that structurally stable heteroclinic cycles can occur in the flow of $\dot{x} = x + Q(x)$.

Proposition 2.2. *Heteroclinic cycles exist in the flow of $\dot{x} = x + Q(x)$ lying in invariant (x_i, x_{i+1})-coordinate planes and joining zeros on the coordinate axes whenever*

$$a < 0 \ ,$$
$$bd < 0 \ .$$

Proof. The conditions are sufficient to show (using the Poincaré-Bendixson Theorem) that in each invariant (x_i, x_{i+1})- coordinate plane, the one-dimensional stable manifolds of the saddle equilibrium points on one of the coordinate axes lie in the two-dimensional stable manifolds of the sink

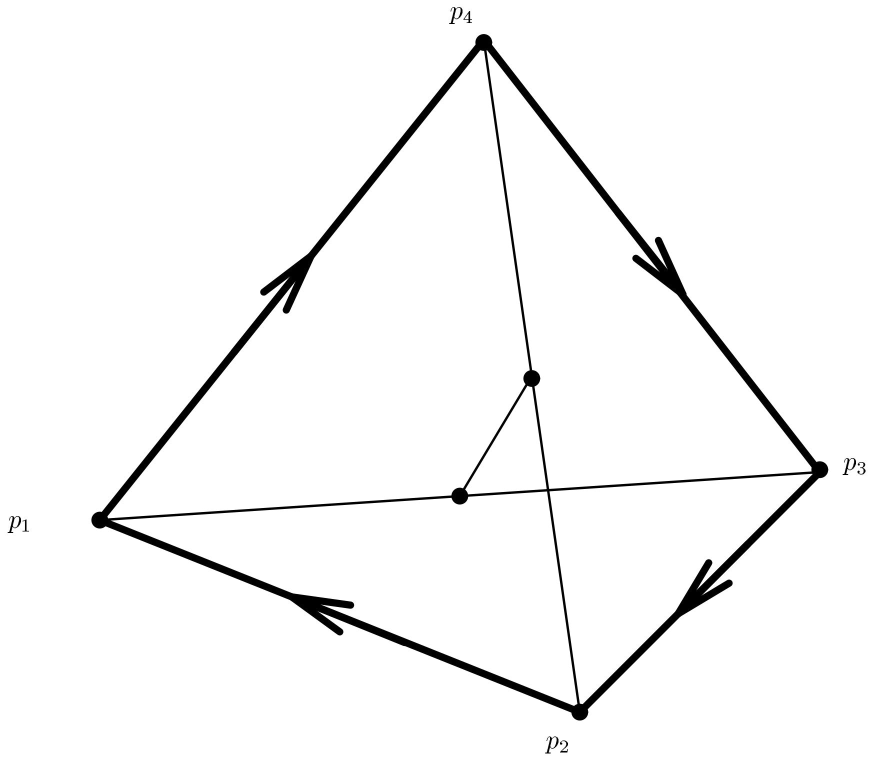

FIGURE 1. A heteroclinic cycle on Δ.

equilibrium points on the other coordinate axis. The images of these connecting orbits under the group action form heteroclinic cycles. $\square$

When Q is contracting, the heteroclinic cycles of Proposition 2.2 are given by the curves which are the intersection of $\mathcal{S}$ with the invariant (x_i, x_{i+1})-coordinate planes. The flow is most easily visualized by considering a "fundamental domain" in the phase space defined as

$$\Delta \equiv \mathcal{S} \cap \{x \;:\; x_i \geq 0\}.$$

The region Δ may be thought of as a tetrahedron where the corners correspond to points in $\mathcal{S}$ on coordinate axes, the edges correspond to points in $\mathcal{S}$ in coordinate planes, and the faces correspond to points in $\mathcal{S}$ in coordinate hyperplanes. The fundamental domain Δ containing a heteroclinic cycle is pictured in Figure 1. The equilibrium points at the corners are labeled so that p_1 corresponds to the equilibrium point on the positive x_1-axis, p_2 corresponds to the equilibrium point on the positive x_2-axis, and similarly for p_3 and p_4. If the flow on Δ is known then the flow on the entire attracting invariant sphere $\mathcal{S}$ is known by application of elements of the symmetry group G.

We shall now discuss in greater detail the flow on the x_1-axis as a representative of the flows in all the coordinate axes and then the flow in the (x_1, x_2)-plane as a representative of the flows in all the (x_i, x_{i+1})-coordinate planes.

Note first that the invariant x_1-axis intersects the invariant sphere $\mathcal{S}$ in a single point which must be an equilibrium point for the flow. Call this point p and note that it corresponds to the point p_1 in Figure 1. The symmetry forces the linearized vector field at p to have four real eigenvalues and eigenvectors which are the coordinate vectors. Associate labels for the eigenvalues at p to their eigenvectors as follows:

$$\lambda_N, (1,0,0,0);\ \lambda_s, (0,1,0,0);\ \lambda_{ss}, (0,0,1,0);\ \lambda_u, (0,0,0,1)\ .$$

The eigenvector corresponding to λ_N is normal to the invariant sphere $\mathcal{S}$, while the others are tangent to the sphere. For the cubic normal form, we easily compute that

$$\lambda_N = -2,\ \lambda_s = d,\ \lambda_{ss} = c, \lambda_u = b\ .$$

As the notation may have indicated, we are interested only when p is a saddle point with a one-dimensional unstable manifold. We shall assume that λ_u is the positive eigenvalue which is associated to the eigenvector $(0,0,0,1)$ and thus the unstable manifold lies in the invariant (x_1, x_4)-coordinate plane. This results in a heteroclinic cycle as pictured in Figure 1.

Now consider the flow in the invariant (x_1, x_2)-coordinate plane, which we shall call $\mathcal{P}$. The invariant sphere $\mathcal{S}$ intersects $\mathcal{P}$ in a (topological) circle that contains the equilibrium point p and three elements of its group orbit. If there are no equilibrium points in $\mathcal{P}$ which do not lie on one of the coordinate axes, then there are connecting orbits from p and $-p$ to the equilibrium points corresponding to p on the the x_2-axis. This is pictured in Figure 2. We label the equilibrium point on the positive x_2-axis with q (corresponding to the point p_2 in Figure 1) and label the orbit connecting p and q with γ. The segment γ is one part of the homoclinic cycle.

Next we specify properties of the flow in a neighborhood of the equilibrium points p. These are expressed by linear relations on the eigenvalues at p. We shall temporarily defer an explanation of why these are necessary for our analysis, so for now simply assume that

$$\lambda_u + 3\lambda_s > 0\ ,\quad \lambda_u + \lambda_{ss} + \lambda_s < 0\ . \tag{4}$$

In view of the earlier calculations, for the cubic normal form this corresponds to $b + 3d > 0$ and $b + c + d < 0$.

So far, we have given conditions for the cubic normal form to contain a particular homoclinic cycle. An analogous cycle exists in any equivariant vector field in $C^r_G(\mathbb{R}^4, \mathbb{R}^4)$ that satisfies the four conditions:

(A1) The flow contains an invariant topological 3-sphere, $\mathcal{S}$. The sphere is attracting, sufficiently smooth, and can be parameterized as a graph over the unit sphere.

(A2) The eigenvalues of the linearization at $p = S \cap \{$the positive x_1-axis$\}$ associated to the eigenvectors by $\lambda_N, (1,0,0,0)$; $\lambda_s, (0,1,0,0)$; $\lambda_{ss}, (0,0,1,0)$; $\lambda_u, (0,0,0,1)$ satisfy

$$\lambda_N, \lambda_s, \lambda_{ss} < 0,\ \lambda_u > 0\ .$$

(A3) $W^u(q) \subset W^s(p) \cap W^s(-p)$, where $q = S \cap \{$the positive x_2-axis$\}$.

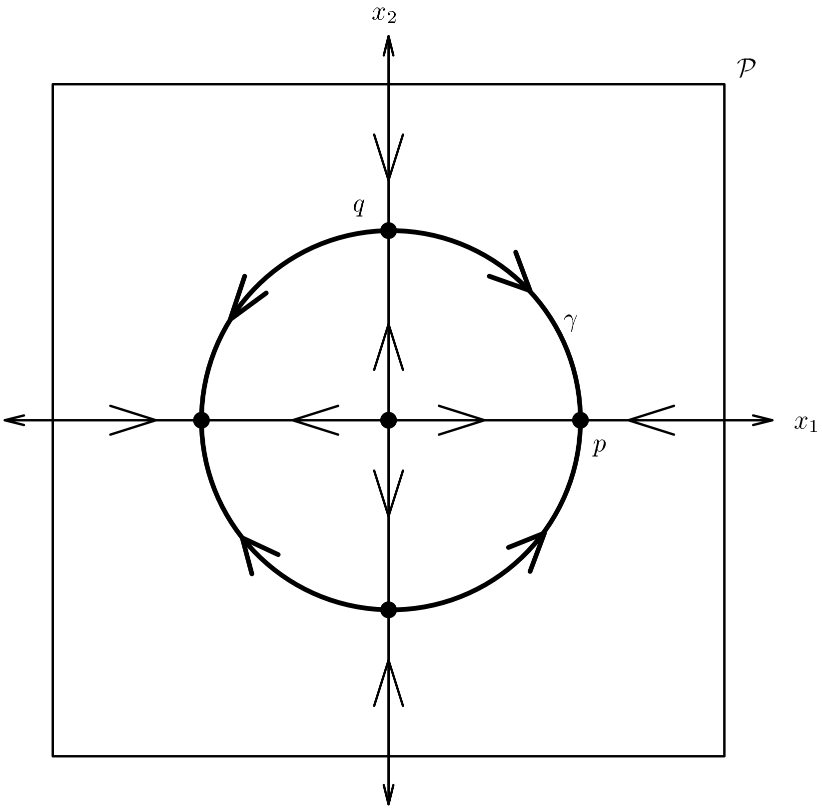

FIGURE 2. The geometry of the flow in $\mathcal{P}$.

(A4) $\lambda_u + 3\lambda_s > 0$, $\lambda_u + \lambda_{ss} + \lambda_s < 0$.

For the cubic vector field, we have already given sufficient conditions on the parameters to know when (A1),...,(A4) hold. In general, we wish to consider vector fields X in the space of C^r_G vector fields with $DX(0) = 1$. Within this space, there is an open set of vector fields that satisfy these four geometric conditions. Note that these are open conditions. As pointed out by one of the referees, if $j_3X(0)$ satisfies the conditions specified for the cubic normal form and $X - j_3X(0)$ is sufficiently small in the appropriate C^r norm on a ball of finite radius (that includes $\mathcal{S}$), then X will also satisfy the conditions. Thus let $\mathcal{M}_0$ be the open set of vector fields X in $C^r_G(\mathbb{R}^4, \mathbb{R}^4) \cap \{DX(0) = 1\}$ that satisfy (A1),...,(A4).

Remarks:

1. Due to the sign constraints on the eigenvalues in (A2), the unstable manifold of p, $W^u(p)$, is one-dimensional, and the stable manifold, $W^s(p)$, is three-dimensional. This is also true for each point in the G-orbit of p, in particular q. From a simple count of dimensions, $W^u(q)$ and $W^s(p)$ cannot intersect transversally, but they do intersect G-transversally so (A3) is an

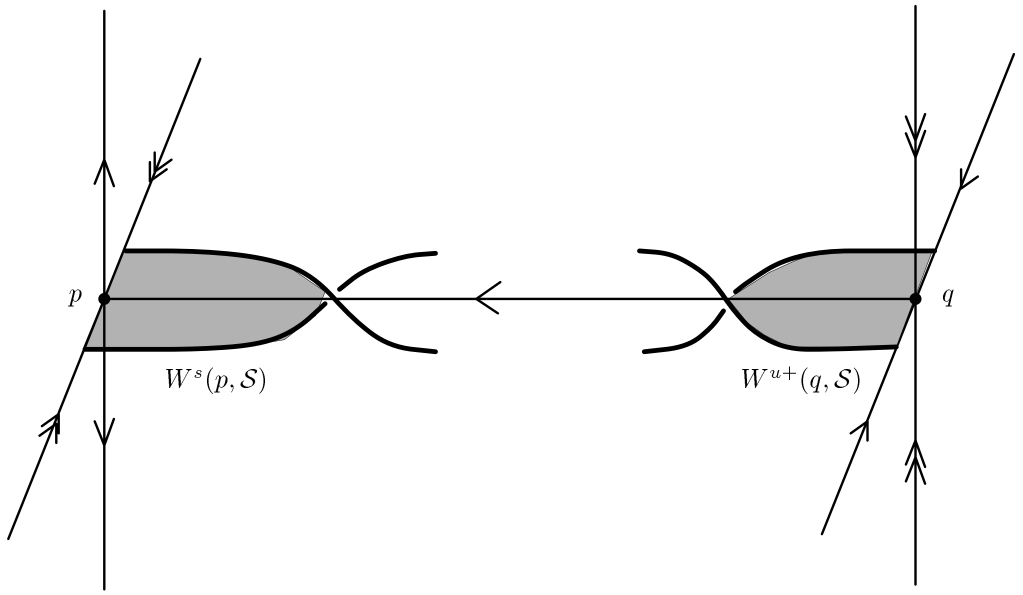

FIGURE 3. A sketch of the local geometry of $W^s(p,\mathcal{S})$ and $W^{u+}(q,\mathcal{S})$.

open condition. The symmetry and (A3) together imply that p is the equilibrium point in a structurally stable homoclinic cycle. For a discussion of G-transversality see [5].

2. The conditions placed on the eigenvalues in (A4) will ensure the appropriate behavior in the neighborhood of the equilibrium points in the cycle and will be discussed later. We note that they imply $\lambda_{ss} < \lambda_s < 0$. Consequently the strong stable manifold of the equilibrium point p for the flow in $\mathcal{S}$ lies in the (x_1, x_3)-coordinate plane.

Following the analysis of Homburg, Kokubu & Krupa [14] we define two invariant manifolds for the flow in $\mathcal{S}$. Let $W^s(p,\mathcal{S})$ be the stable manifold of the point p with respect to the flow in $\mathcal{S}$. It is two-dimensional, contains γ, and at p is tangent to the eigenvectors $(0,1,0,0)$ and $(0,0,1,0)$ (since $\lambda_{ss} < \lambda_s$). Let $W^{u+}(q,\mathcal{S})$ be an invariant manifold in $\mathcal{S}$ which is tangent at q to the eigenvectors $(1,0,0,0)$ and $(0,0,1,0)$. This manifold is called the extended unstable manifold of q since it is tangent at q to all the unstable directions plus the weakest stable direction. It is two-dimensional and contains γ, but it is not unique. However, its tangent space along γ is unique. In Figure 3 we indicate the local geometry of $W^s(p,\mathcal{S})$ and $W^{u+}(q,\mathcal{S})$.

Let $\mathcal{M}_1 \subset \mathcal{M}_0$ be the subset of vector fields for which $T_\gamma W^s(p,\mathcal{S}) = T_\gamma W^{u+}(q,\mathcal{S})$, *i.e.*, the vector fields for which the stable manifold of p and the extended stable manifold of q are tangent along γ. Tangency at a single point on γ implies tangency everywhere on γ. This condition is the same as the twist degeneracy condition for homoclinic orbits [3, 12, 14]. These orbits have been called *inclination-flip* homoclinic orbits, so we adopt this term also for homoclinic cycles with a twist degeneracy. We claim, but do not show, that $\mathcal{M}_1$ is a C^{r-1} submanifold of $\mathcal{M}_0$ of codimension one.

We will now consider parameterized families of equivariant vector fields which have the geometric properties we have been describing in (A1),...,(A4). Thus assume

(A5) $X_\mu \in \mathcal{M}_0$ for all $\mu \in U$, where U is an open subset of $\mathbb{R}^n$.

We also assume that there exists a bifurcation point where the heteroclinic orbit has a twist degeneracy. This is assumption

(A6) There exists $\mu_0 \in U$ such that $X_{\mu_0} \in \mathcal{M}_1$ and the mapping $\mu \mapsto X_\mu$ is transversal to $\mathcal{M}_1$ at μ_0.

We need to make a nondegeneracy assumption at the bifurcation point. This is simply that the tangency of the two manifolds $W^s(p,\mathcal{S})$ and $W^{u+}(q,\mathcal{S})$ along γ is generic. Thus we assume

(A7) In the flow of X_{μ_0}, the manifolds $W^s(p,\mathcal{S})$ and $W^{u+}(q,\mathcal{S})$ have order one contact along γ.

This is an open condition, but if it does not hold, then an arbitrarily small perturbation to the vector field within $\mathcal{M}_1$ can make it non-zero. This is analogous to the idea behind the argument that vector fields with homoclinic connections can be perturbed an arbitrarily small amount to break the connection. This means that vector fields within $\mathcal{M}_1$ satisfy (A7) generically. In fact, the symmetry forces any positive contact to have odd order and all we will need is finite order. In general, we must be a little careful since W^{u+} is not unique and its smoothness depends on the ratio λ_{ss}/λ_s. However, the next assumption we make eliminates any difficulties arising from this.

Finally, to simplify the analysis, we will assume that we can equivariantly linearize the vector field in a neighborhood of p. We state this condition now:

(A8) There is a neighborhood of $p \in \mathbb{R}^4$ and $\mu_0 \in U$ which is uniformly C^3 equivariantly linearizable.

This means that there is a family of C^3 diffeomorphisms that linearize X_μ in a neighborhood of p while preserving the equivariance. This family of diffeomorphisms varies smoothly with μ. Sufficient conditions for (A8) to hold are given by Rychlik [17, Theorem 1.1]. The linearizability question is a perennial problem in the theory of dynamical systems [1]. For our purposes we will state simply that (A8) does not hold when there are low order resonances between the eigenvalues at the equilibrium point. For any set of eigenvalues for which (A8) does not hold, there is another set close by for which (A8) will hold. We can now define $W^{u+}(q,\mathcal{S})$ uniquely as the extended unstable manifold which is linear in a neighborhood of q.

The main result of this paper is stated in the following theorem which we shall prove in the following section.

Theorem 2.3. *Let X_μ be a family of vector fields satisfying (A5)-(A8). Then there are values of μ arbitrarily close to μ_0 such that the flow associated with X_μ contains hyperbolic invariant chaotic sets.*

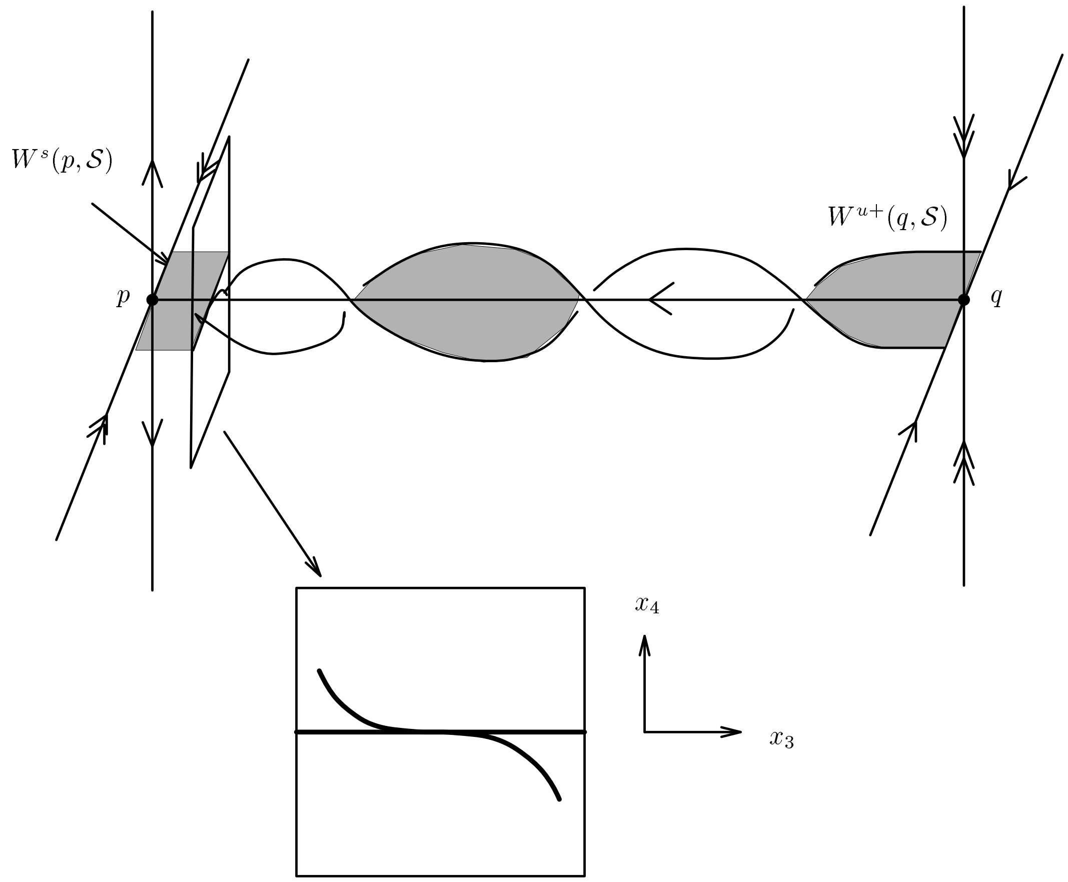

FIGURE 4. Model of locally linearized flow.

3. THE CONSTRUCTION OF A POINCARÉ RETURN MAP

We prove Theorem 2.3 by constructing a Poincaré map and demonstrating that this map contains hyperbolic chaotic invariant sets (horseshoes). This section is devoted to the construction and analysis of the map.

Suppose that X_μ is a family satisfying (A5)-(A8) and let $\tilde{X}_\mu$ be the associated family that exists due to (A8) for which the flow is linear in a neighborhood V of the equilibrium point p for some neighborhood of μ_0. We work with the family $\tilde{X}_\mu$ since the dynamics are the same, *i.e.*, the flows generated are C^3 equivalent. We shall assume without loss of generality that the flow at p is linear in a neighborhood of the unit cube at p and then construct maps between boundaries of the unit cube. Furthermore we restrict ourselves to the flow within $\mathcal{S}$, the invariant topological 3-sphere. See Figure 4.

In the linear region near $p = (p_1, 0, 0, 0)$, lying in $\mathcal{S}$ is equivalent to setting $x_1 = p_1$. We define the cross sections, $\Sigma_0 \equiv \{(p_1, 1, x_3, x_4) : |x_3|, |x_4| < 1\}$ and $\Sigma_1 \equiv \{(p_1, x_2, x_3, 1) : |x_2|, |x_3| < 1\}$.

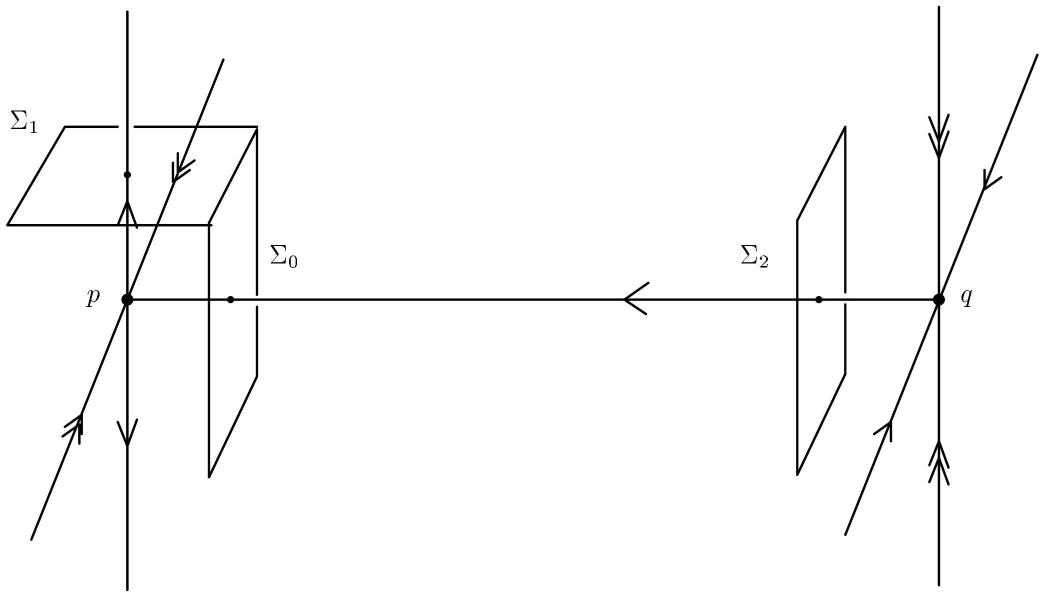

FIGURE 5. The sections Σ_0, Σ_1, and Σ_2.

See Figure 5. Let L_μ be the map from Σ_0 to Σ_1 induced by the flow. This map is singular and is computed by explicitly integrating the linear flow. So $L_\mu : \Sigma_0 \to \Sigma_1$ is given by

$$(p_1, 1, x_3, x_4) \mapsto (p_1, x_4^\alpha, x_3 x_4^\beta, 1) \tag{5}$$

where

$$\alpha = -\lambda_s/\lambda_u > 0,\ \beta = -\lambda_{ss}/\lambda_u > 0.$$

For a point on Σ_0 with $x_4 < 0$, the linear flow brings it to the surface $\hat{\Sigma}_1 = \{(p_1, x_2, x_3, -1) : |x_2|, |x_3| < 1\}$. However, $\hat{\Sigma}_1$ may be mapped to Σ_1 by an element of G. The resulting map may be extended to be both continuous and invertible across $x_4 = 0$ and is the map specified by Equation 5.

Define $\Sigma_2 = g\Sigma_1$ to be a cross section in the neighborhood of q where we are assuming the flow is linear. We reduce by the symmetry by considering the composition $g \circ L_\mu$ which maps Σ_0 to Σ_2 with g defined by

$$(p_1, x_2, x_3, 1) \mapsto (1, p_1, x_2, -x_3)\ .$$

We note that this map is only unique up to a rotation by π in the final two coordinates. In other words we have a choice as to which coordinate needs a minus sign. This freedom is eliminated if we consider the full quotient space $\mathbb{R}^4/G$.

Let Ψ_μ denote the map from Σ_2 to Σ_0 induced by the flow $\tilde{X}_\mu$. We assume that Ψ_μ is a family of C^3 diffeomorphisms and it will be approximated by a Taylor series. The parameter μ will be varied to produce the heteroclinic bifurcation. The heteroclinic cycle has a symmetry since points on the cycle have a nontrivial isotropy subgroup. This symmetry persists in the maps Ψ_μ with the requirement that $\Psi_\mu(-x_2,-x_3) = -\Psi_\mu(x_2,x_3)$. The map $\Psi_\mu : \Sigma_2 \to \Sigma_0$ may be defined by

$$(1,p_1,x_3,x_4) \mapsto (p_1,1,\psi_3(x_3,x_4),\psi_4(x_3,x_4))$$

or, by neglecting the first two terms, as

$$(x_3,x_4) \mapsto (\psi_3(x_3,x_4),\psi_4(x_3,x_4))$$

with $\psi_i(-x_3,-x_4) = -\psi_i(x_3,x_4)$. Note that we did not express the explicit dependence on the parameter μ for notational convenience.

We write the derivative of the map Ψ_μ at $(0,0)$ as

$$D\Psi_\mu|_{(0,0)} = \begin{pmatrix} \eta_{31}(\mu) & \eta_{32}(\mu) \\ \eta_{41}(\mu) & \eta_{42}(\mu) \end{pmatrix} .$$

Condition (A6) says that

$$D\Psi_{\mu_0}|_{(0,0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} ,\ c_1 \neq 0 ,$$

implying that $\eta_{31}(\mu_0) \neq 0$ and $\eta_{41}(\mu_0) = 0$. The invertibility of Ψ_μ further requires that $\eta_{42}(\mu_0) \neq 0$. We may smoothly reparameterize the family $\tilde{X}_\mu$ and write using a Taylor series expansion

$$\Psi_\mu = \begin{pmatrix} \eta_{31}x_3 + \eta_{32}x_4 \\ (\mu-\mu_0)x_3 + Ax_3^3 + \eta_{42}x_4 \end{pmatrix} + \text{ h.o.t. } .$$

The term Ax_3^3 has been introduced since we wish to study this map near $\mu = \mu_0$ and require that the higher order terms all be dominated by the given terms when x_3, x_4 are sufficiently small. The coefficient A is derived from $\frac{\partial^3\Psi_4}{\partial {x_3}^3}|_{(0,0)}$. Condition (A7) is exactly the condition that $A(\mu_0) \neq 0$. We see this Taylor series term is nonzero because $W^{u+}(q,\mathcal{S})$ intersects Σ_2 along the parameterized curve $(1,p_1,s,0)$ for $s \in (-1,1)$. Consequently $W^{u+}(q,\mathcal{S}) \cap \Sigma_0$ is given by the curve $(p_1,1,\Psi_3(s,0),\Psi_4(s,0))$. A straightforward calculation shows that the tangency condition in (A7) is given by $\frac{d\Psi_4}{d\Psi_3}|_{s=0} = \frac{{\Psi_4}'''}{{\Psi_3}'}|_{s=0} \neq 0$ where the prime denotes differentiation with respect to s. So ${\Psi_4}'''|_{s=0} \neq 0$ and ${\Psi_4}'''|_{s=0} = \frac{\partial^3\Psi_4}{\partial {x_3}^3}|_{\mu_0,(0,0)}$. The geometrical significance of the nonzero cubic term is that the intersection $W^{u+}(q,\mathcal{S}) \cap \Sigma_0$ is qualitatively like the graph from Figure 4 with a cubic tangency.

The composition of the linear map $g \circ L_\mu$ and the return map Ψ_μ gives the family $P_\mu : \Sigma_0 \to \Sigma_0$ defined by $(x_3,x_4) \mapsto \Psi_\mu \circ g \circ L_\mu(x_3,x_4)$ and given by

$$P_\mu \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \eta_{31}x_4^\alpha - \eta_{32}x_3x_4^\beta \\ (\mu-\mu_0)x_4^\alpha + Ax_4^{3\alpha} - \eta_{42}x_3x_4^\beta \end{pmatrix} + \text{ h.o.t. } .$$

This family of maps models the flow in a neighborhood of the homoclinic cycles. We shall proceed with an analysis of a family of maps of this type.

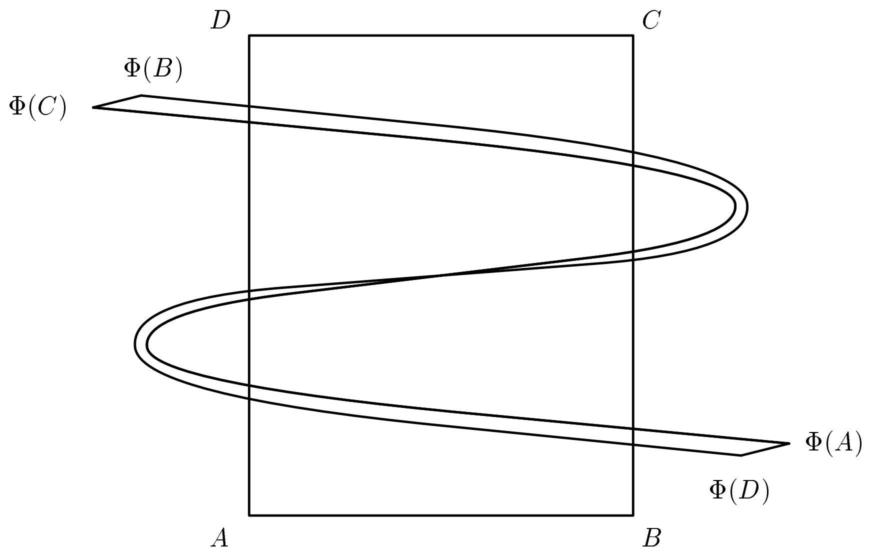

FIGURE 6. The image of the rectangle $R_\mu = ABCD$ under Φ.

By rescaling the map P_μ, changing variables and dropping higher order terms, one gets the nonlinear map:

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu x^\alpha - x^{3\alpha} + x^\beta y \\ b_1 x^\alpha + b_2 x^\beta y \end{pmatrix} . \tag{6}$$

The only assumption necessary to perform these calculations was that $A \neq 0$. Conditions (A2) and (A4) restrict the values of α and β to the following ranges:

$$0 < \alpha = -\lambda_s/\lambda_u < 1/3, \ 1 - \alpha < \beta = -\lambda_{ss}/\lambda_u \ .$$

In fact (A4) was chosen so that α and β would satisfy these inequalities. In the following theorem, we show that this mapping has a complicated invariant set for values of α and β in this range. This gives us the necessary behavior for the family $\tilde{X}_\mu$ near $\mu = \mu_0$ and finishes the proof of Theorem 2.3.

Theorem 3.1. *Assume that $0 < 3\alpha < 1$ and $1 - \alpha < \beta$ then the mapping Φ defined by Equation 6 contains a hyperbolic chaotic invariant set for $\mu > 0$, but sufficiently small.*

Proof. We study this mapping on the rectangle $R_\mu = [-2\mu^{1/2\alpha}, 2\mu^{1/2\alpha}] \times [-\mu^\gamma, \mu^\gamma]$ (where $\gamma > 0$ is still to be determined) and are interested in the limit as $\mu \to 0$. This map may be thought of as stretching and folding as pictured in Figure 6.

Rescaling under the singular transformation given by $z = \mu^{-1/2\alpha} x, w = \mu^{-\gamma} y$, the mapping becomes

$$\hat{\Phi} \begin{pmatrix} z \\ w \end{pmatrix} = \mu^{(3\alpha-1)/(2\alpha)} \begin{pmatrix} z^\alpha - z^{3\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu^{\gamma+(\beta-1)/2\alpha} z^\beta w \\ \mu^{\beta/2\alpha} b_1 z^\beta w + \mu^{-\gamma+1/2} b_2 z^\alpha \end{pmatrix} \tag{7}$$

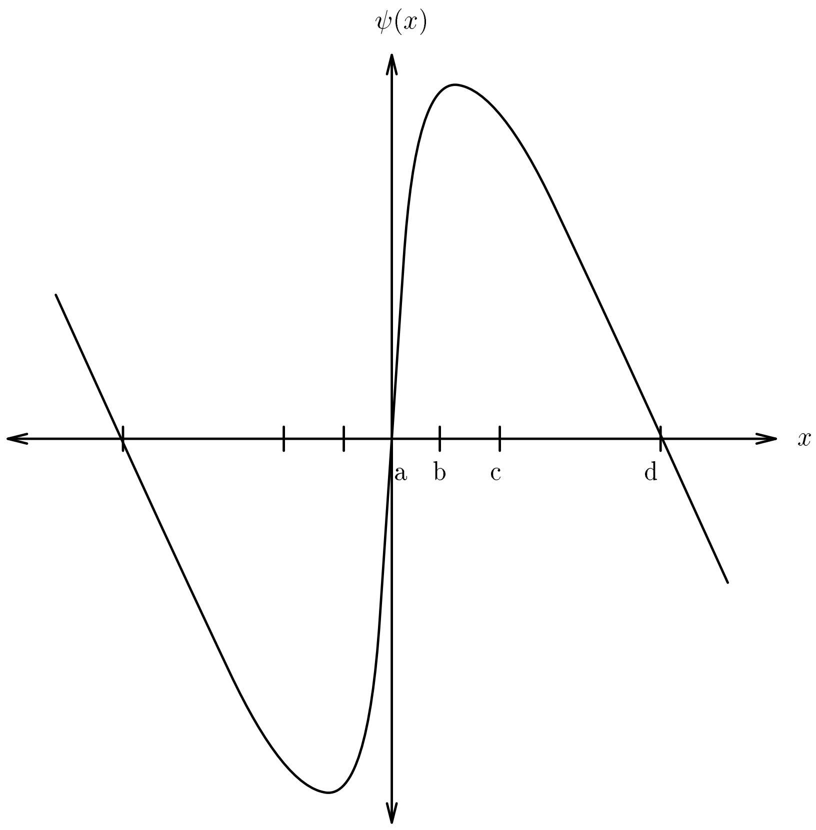

FIGURE 7. The one-dimensional map $\psi(x)$.

and $\hat{R}_\mu = [-2,2] \times [-1,1]$. Observe that since $(3\alpha - 1)/2\alpha < 0$ that the first term in the right hand side of Equation 7 is singular with respect to μ. Requiring that $\gamma \in ((1-\beta)/2\alpha, 1/2)$ makes all powers of μ in the second term of the right hand side positive. So this mapping converges in a singular limit to a rescaling of the rank one map given by

$$\Psi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu x^\alpha - x^{3\alpha} \\ 0 \end{pmatrix} .$$

We have reduced the problem to the study of a one-dimensional map which comes from the first component of Ψ,

$$\psi(x) = \mu x^\alpha - x^{3\alpha} .$$

A graph of $\psi(x)$ is given in Figure 7.

Consider the points

$$a = 0,\ b = (\mu/4)^{1/2\alpha},\ c = (\mu/2)^{1/2\alpha} \text{ and } d = \mu^{1/2\alpha}.$$

Assuming that μ is small, we have $\psi(a) = \psi(d) = a$ and $\psi(b) = 3(\mu/4)^{3/2} \gg d$ and $\psi(c) = (\mu/2)^{3/2} \gg d$. Further, $\psi' > 1$ on $[a, b]$ and $\psi' < 1$ on $[c, d]$. So ψ has a hyperbolic invariant set in the interval $[a, d]$. The dynamics on this set are chaotic and are well known from the study of the logistic map. See, for example, [4]. Due to the symmetry, there will be identical behavior for $x < 0$. The interaction between the negative and positive humps of the map produce additional invariant sets.

We conclude that the hyperbolic invariant set of the singular limit will persist as horseshoes in Φ when we leave the singular limit. This is analogous to the behavior of the Hénon map in the limit as the map tends to rank one [13]. $\square$

4. The analytic problem

In the previous section we constructed a geometric model for a G-equivariant homoclinic bifurcation. We would like to prove that this bifurcation occurs in the cubic normal form for G-equivariant vector fields given in Equation 3. This would then give a proof for the existence of a direct bifurcation from an attracting fixed point to a chaotic invariant set. The primary difficulty, and one which we have not resolved, is in proving that within the four-parameter family there is a one-parameter family which satisfies the nondegeneracy condition expressed in (A7). However, we can prove the following theorem about the family of normal forms:

Theorem 4.1. *There are open sets of values for b, c, d such that there exists a countably infinite number of functions $\tilde{e}_i(b, c, d)$, $i \in Z$, such that the flow of $\dot{x} = x + Q(x, -1, b, c, d, e)$ contains an inclination-flip heteroclinic orbit if and only if $e \in \{\tilde{e}_i(b, c, d)\}_{i \in Z}$.*

We conjecture that a countably infinite number of these bifurcation points will be nondegenerate. The bifurcation at $e = \tilde{e}_0(b, c, d) \equiv 0$ is degenerate. We proceed to prove this theorem by examining the heteroclinic cycles in the normal form and attempting to find conditions on the parameter values for which the normal form satisfies assumptions (A5)-(A8) of the previous section.

As already noted in the previous section, the normal form $\dot{x} = x + Q(x, -1, b, c, d, e)$ has eigenvalues of $\lambda_N = -2, \lambda_u = b, \lambda_{ss} = c, \lambda_s = d$ at p. Therefore, it satisfies (A1)-(A4) when $Q(x)$ is contracting, $b > 0$, $b + 3d > 0$, and $b + c + d < 0$ and b, c, d are scaled small enough to give sufficient smoothness of $\mathcal{S}$ in a neighborhood of the heteroclinic cycle. The twist conditions (A5) and (A6) can be satisfied and we examine this now.

In order to study the twist around the homoclinic cycle, we first develop an approximation to the cycle. It lies in the coordinate planes and as b, c, d approach zero, must approach one quarter of a circle. We therefore scale the parameters b, c, d by $(b, c, d) \mapsto (\epsilon b, \epsilon c, \epsilon d)$ and perform a perturbation analysis assuming that $\epsilon \ll 1$. The equations in the $x_1 - x_2$ plane reduce to

$$\begin{aligned} \dot{x_1} &= x_1(1 - x_1^2 - x_2^2 + \epsilon b x_2^2) \\ \dot{x_2} &= x_2(1 - x_1^2 - x_2^2 + \epsilon d x_1^2) \; . \end{aligned}$$

Define $\alpha = (b+d)/4$ and $\beta = (b-d)/4$, then writing the above vector field in polar coordinates gives

$$\begin{aligned} \dot{r} &= r(1-r^2) + \epsilon \alpha r^3 \sin^2 2\theta \\ \dot{\theta} &= \epsilon r^2 \sin 2\theta(\alpha \cos 2\theta - \beta) \ . \end{aligned}$$

Since $\beta > \alpha > 0$, we see that $\dot{\theta} < 0$ for $\theta \in (0, \pi/2)$. For small $\epsilon > 0$ there exists a connecting orbit from $(r, \theta) = [1, \pi/2]$ to $[1, 0]$ close to the curve defined by $r = 1$. We perform a perturbation analysis to study this curve. We wish to parameterize the curve by $(g(\theta), \theta)$ and write

$$g(\theta) = 1 + \epsilon g_1(\theta) + \epsilon^2 g_2(\theta) + O(\epsilon^3).$$

Setting $\dot{r} = \frac{dg(\theta)}{d\theta}\dot{\theta}$, we can equate powers of epsilon and solve for g_1 and g_2. This gives

$$g(\theta) = 1 + \epsilon\frac{\alpha}{2} sin^2 2\theta + \epsilon^2 \frac{\alpha}{2} sin^2 2\theta(5\alpha \sin^2 2\theta - (3+4\alpha)/4 + 2\beta \cos 2\theta) + O(\epsilon^3).$$

To examine the twist around the cycle, we transform x_3 and x_4 into polar coordinates (s, ϕ). The differential equation for $\phi \in S^1 \equiv \mathbb{R} \bmod 2\pi$ becomes

$$\begin{aligned} \dot{\phi} &= -\tfrac{e}{2}\sin 2\theta + s^2 \tfrac{e}{2}\sin 2\theta + \epsilon \sin 2\phi(\tfrac{b}{2}\cos^2\theta - \tfrac{c}{2}\cos 2\theta - \tfrac{d}{2}\sin^2\theta) + \\ & \quad \epsilon s^2 \sin 2\phi(-\tfrac{b}{2}(\cos^2\theta - \sin^2\phi) + \tfrac{c}{2}\cos 2\theta + \tfrac{d}{2}(\sin^2\theta + \cos^2\phi)) \ . \end{aligned}$$

The twist about the cycle is given by integrating the above equation with $s = 0$ along the homoclinic cycle. Although we cannot do this explicitly, the results we need follow from studying the flow of θ and ϕ on the cylindrical section $(\theta, \phi) \in [0, \pi/2] \times S^1$. The flow is given by the equations

$$\begin{aligned} \dot{\theta} &= \epsilon g^2(\theta) \sin 2\theta(\alpha \cos 2\theta - \beta) \\ \dot{\phi} &= -\tfrac{e}{2}\sin 2\theta + \epsilon \sin 2\phi(\tfrac{b}{2}\cos^2\theta - \tfrac{c}{2}\cos 2\theta - \tfrac{d}{2}\sin^2\theta) \ . \end{aligned}$$

This flow has equilibrium points which are saddles at $(0,0)$, $(0,\pi)$, $(\pi/2, 0)$, $(\pi/2, \pi)$, sinks at $(0, \pi/2)$, $(0, 3\pi/2)$, and sources at $(\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$. The twist condition (A6) corresponds to a connecting orbit between the saddles $(\pi/2, 0)$ and $(0,0)$. This is most easily understood by considering the geometry as pictured in Figure 8. We know that θ decreases monotonically from $\pi/2$ to 0 and note that $\dot{\phi}$ decreases pointwise as e is increased. This means that as we increase e, the number of times that the unstable manifold of $(\pi/2, 0)$ twists around the cylinder increases and there will be a countably infinite number of values of $e = e_0(\epsilon b, \epsilon c, \epsilon d)$ for which the desired connection occurs.

We now turn to condition (A5) and the remainder of (A6). Taking a one-parameter family of normal forms which is transversal to $\mathcal{M}_1$ is accomplished by considering e to be the distinguished parameter. This follows directly from consideration of the vector field on the cylindrical section again. Perturbing e from a value at which a connecting orbit exists will break the orbit, so our family is transversal to $\mathcal{M}_1$.

The eigenvalues at p are given by $-2, b, c, d$ and do not vary with e. It is therefore easy to choose them with no low order resonances so that the normal form will be linearizable in a neighborhood

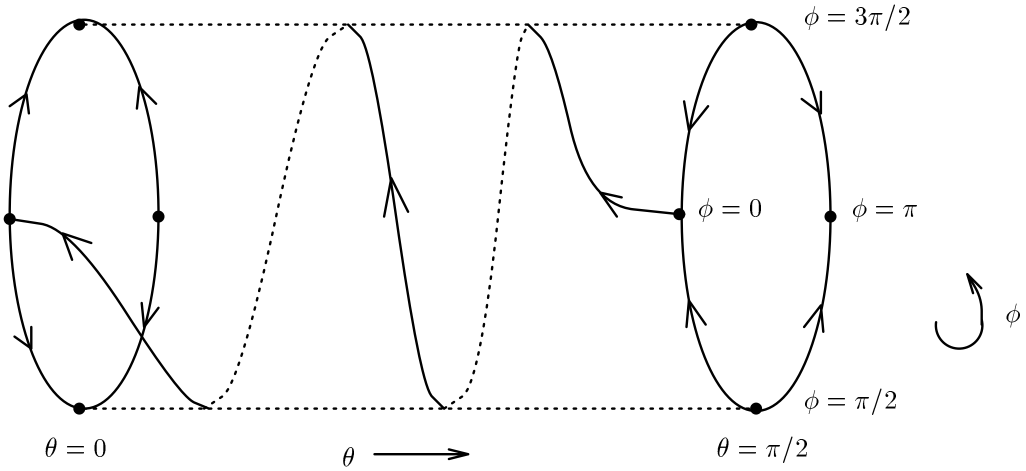

FIGURE 8. A connecting orbit in the flow representing twisting about the homoclinic cycle.

of p, satisfying condition (A8). Further, we may insist that for all integer vectors Q,

$$|\,(Q,(-2,b,c,d))\,| > \epsilon |Q|^{-\nu}$$

where $|Q| = \sum_{i=1}^{4} |q_i|$ for some $\nu > 0, \epsilon > 0$. This condition holds for almost all values of b, c, d with respect to the Lebesgue measure and gives the existence of a linearizing change of coordinates which is analytic at p [2, 18]. Consequently, we may choose linearizing coordinates as smooth as necessary so that the manifolds $W^s(q,\mathcal{S})$ and $W^{u+}(p,\mathcal{S})$ are well defined and smooth enough for our calculations. We do not have a good way to approximate these manifolds to determine whether the normal form contains degenerate inclination-flip heteroclinic cycles. We cannot prove that the normal form family with critical twist does not lie in the submanifold of $\mathcal{M}_1$ that contains vector fields with higher order tangencies, thus satisfying (A7).

We have performed some numerical calculations designed to examine the degeneracy of the twist tangency in the normal form. We observe that the twisting about the heteroclinic cycle appears to be nonlinear and has the same properties as the global flow of the geometrical model. These calculations indicate that the tangency is indeed nondegenerate, but of course, due to the sensitivity of the situation, consist of far less than a proof.

5. Conclusions

Symmetries arise naturally in many simple mathematical models. The variety of behavior which typically occurs is frequently model independent — it depends largely on the symmetry. The specific behavior depends on the actual parameters in the model, but the possibilities can be independently studied. Equivariant bifurcation theory is aimed at classifying the different generic bifurcations with symmetry and uses techniques from group theory and singularity theory. We have exhibited a specific symmetry group whose equivariant bifurcations can have extraordinary complexity.

As well as altering the generic local bifurcations, the presence of symmetry affects many global structures. Direct global analysis of the normal form seems difficult, instead we have examined a geometrical model which shares the same symmetry group. This equivariant family possesses structurally stable heteroclinic cycles for large parameter sets. In a one-parameter family, these heteroclinic cycles can undergo a global bifurcation which does not break them, but the flow around them twists degenerately. This bifurcation point corresponds to an inclination-flip codimension two point for an asymmetric homoclinic orbit. For an open set of parameter values in a neighborhood of the bifurcation point there are Smale horseshoes embedded in the flow and the heteroclinic cycles still exist. We believe that this bifurcation occurs in the cubic normal form of the instant chaos bifurcation.

This is one example of a homoclinic/heteroclinic bifurcation as a mechanism for the generation of chaos. It is a common phenomenon for homoclinic and heteroclinic orbits to organize the dynamics of a nonlinear system. This is especially true in many symmetric systems where the heteroclinic cycles can be structurally stable. In the instant chaos equivariant normal form, not only do the the heteroclinic cycles play an important role in the dynamics, but as the numerical evidence presented indicates, they may be involved in a one-parameter bifurcation which creates chaotic hyperbolic invariant sets in the flow.

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