The geometry of halo orbits in the circular restricted three-body problem

Robert Thurman and Patrick A. Worfolk

The Geometry Center
University of Minnesota
1300 S. Second St., Suite 500
Minneapolis, MN 55454

October 25, 1996

Abstract

In the summer of 1996, we supervised two undergraduate students during a nine-week summer program at the Geometry Center. They worked on a project using dynamical systems techniques to compute and visualize orbits in the circular restricted three-body problem. This project was motivated by recent interest in the space science community to send missions near to the Sun-Earth libration points. A fuller understanding of the geometry of the phase space of the circular restricted three-body problem could provide new possibilities for baseline trajectory design. To this end, the goal of this project was to develop computational and visualization tools to aid in trajectory design. In particular, we wanted to be able to easily and interactively explore the geometry of the halo orbits and their stable and unstable manifolds. This report provides a summary of the mathematics underlying the project and a brief discussion of the results.

Keywords: restricted three-body problem, halo orbits, (un)stable manifolds.

Contents

1 Introduction

For six consecutive years, the Geometry Center has sponsored a Summer Institute for undergraduate students, who work individually or in small groups under the direction of Geometry Center staff members. In the summer of 1996, we supervised two students for the nine-week period of the program on a dynamical systems project. Essentially, the focus of the project was to use dynamical systems techniques to compute and visualize new spacecraft trajectories and was motivated by our interaction with mathematicians at NASA's Jet Propulsion Laboratory. This report provides the mathematical background that we provided our students as well as a brief discussion of their results. An online HTML document [9] created by our students further illustrates these results, in addition to describing the software tools they developed.

Recently, the space science community has shown considerable interest in missions which take place in the vicinity of the libration points of the Sun-Earth system. Designing trajectories for these missions is challenging because conic approximations (solutions of the two-body problem) are inadequate and, in the past, manual numerical searches have been the only recourse. Recent work of Barden, Howell, and Lo [3] has shown that a greater understanding of the dynamics of the restricted three-body problem could lead to innovative baseline mission concepts. Dynamical systems theory could provide insights into the qualitative nonlinear behavior. Knowledge of the geometry of phase space and the existence of stable and unstable manifolds which both separate regions of space and provide natural transfer mechanisms can all aid in trajectory design. Once a preliminary trajectory design has been accomplished within the framework of the restricted three-body problem, the final solution is computed using a model that incorporates ephemeris data and solar radiation pressure.

The primary goal of our project was to develop computational and visualization capabilities for the study of trajectories near the libration points in the restricted three-body problem. This goal was to be accomplished by extending the capabilities of the dynamical systems software package DsTool [2] and using the Geometry Center software Geomview [12] for the visualization. We wanted to create a software environment where it was easy to interactively explore and visualize the dynamics of the restricted three-body problem. Thus for the summer, our first goals were to provide such an environment for the computation of the libration points and the symmetric halo (periodic) orbits surrounding the collinear libration points. Our "championship goals" were to compute and visualize the stable and unstable manifolds of these halo orbits with an emphasis on being able to discover new trajectories which would transfer a spacecraft naturally from a parking orbit about the Earth to the vicinity of one of the libration points.

This paper contains the basic mathematics needed to accomplish the goals outlined above. Most of the mathematics is generally well-known, though it is sprinkled throughout the literature. We have thus tried to bring together the basic ideas, in a manner which is accessible to undergraduates. The details are sketchy in many parts, but all the ideas we found necessary to formally convey to our students have been included. We have ignored the long history of the study of the restricted three-body problem which is covered quite completely by Szebehely [17]. We have also failed to give a complete description of Lagrangian and Hamiltonian mechanics and any formal introduction to dynamical systems.

An overview of the contents is as follows. In Sections 2 and 3, we discuss the central force problem and the two-body problem. From here, we derive the equations of motion for the circular restricted three-body problem in Section 4. In the next section we calculate the libration points and then we present the Jacobian integral. Following this we discuss the symmetries of the equations of motion, since we will take advantage of these in our numerical calculations. In Section 8, we present the Lagrangian formulation of the equations of motion, and in the following section, the Hamiltonian formulation. We present Richardson's technique for analytically approximating halo orbits in Section 10, including many of the gory details not included in his published work. In working through this, we discovered an error in his calculations and present the corrected version here. Following this is a section on the numerical computation of periodic orbits. In particular, an algorithm is given for the computation of the symmetric halo orbits in which we are interested. In Section 12 we present an efficient method for computing the monodromy matrix of the periodic halo orbits. Following this we give a naive, yet effective, algorithm for approximating the stable and unstable manifolds of the halo orbits. Finally, we discuss some of the results of our students' work and point out a variety of interesting questions which we were unable to resolve in the nine-week period.

We found it useful to provide our students (and ourselves) with a number of exercises as a means of "getting our hands dirty" with the background material. We have left these exercises embedded in the document. Solutions are given in Appendix D.

Acknowledgments. Motivation for this project originated from discussions we had with scientists and mathematicians at the Jet Propulsion Laboratory. Martin Lo, in particular, has been instrumental in pushing for a dynamical systems approach to spacecraft trajectory design, and our thanks go to him for both inspiring the project, and providing us with the known techniques for computing libration point halo orbits.

The great success of this project was due primarily to the work of our students, Molly Megraw and Christopher Sinclair. It was a joy to stand back and watch them take off with the project after we provided them with the basic background material. The striking pictures of manifolds that appear in this document were created with their software.

2 The central force problem

Newton was the first to discover that the gravitational force due to a massive central object is one example of a force vector field whose magnitude is inversely proportional to the square of the distance to the object, and directed towards it: $F(X) = -kX/|X|^3$, where $X = (x, y, z)$, and the object is located at the origin. In such a case, F has a *potential function* $V : \mathbf{R^3} \to \mathbf{R}$ given by $V(X) = -k/|X|$, so that $F = -\nabla V$.

Exercise 2.1. Verify this.

The equations of motion for a particle of unit mass under the influence of F are given by Newton's second law, which says that $F = ma$. With $m = 1$ and $a = \ddot{X}(t)$, we have

$$\ddot{X}(t) = -kX/|X|^3.$$

This is a 6 dimensional system; a solution is uniquely determined by six initial conditions – 3 for position and 3 for velocity.

Exercise 2.2. Write down the equations of motion as a system of 6 first-order equations.

The motion is planar

It shouldn't be too surprising that the motion of an object in a central force field stays in a plane, that spanned by the object's initial position and velocity vectors.

Exercise 2.3. Show this. To do so, compute

$$\frac{d}{dt}(X(t) \times \dot{X}(t))$$

for a solution $X(t)$. Recall that the normal product rule holds for the derivative of cross-products. Now think geometrically! The vector $mX \times \dot{X}$ for a mass m is the angular momentum vector. Explain the phrase "angular momentum is conserved" for this motion.

Circular solutions

It can be shown that the path of the object is not only planar but in fact traces out a conic section (see, for example, [5, Chapter 1, Section 1.5]). The proof is not difficult, but involves changing to polar coordinates along with a little trickery. For our purposes, we will just verify that a circular trajectory is indeed one possible solution.

Exercise 2.4. Verify that $(a \cos \omega t, a \sin \omega t, 0)$ is a solution. How does the angular velocity ω depend on a and k?

An important aside: The discovery of conics as solutions involves changing to polar coordinates (r, ϕ) in the plane of motion, and then transforming the equations of motion and eliminating time as the dependent variable. The resulting equation can be solved explicitly for r in terms of ϕ. Thus it is possible to deduce from initial conditions precisely which conic section path an object under the influence of a central force will follow. It is *not* in general possible, however, to find an explicit *parameterization*, using elementary functions, of the path with time as the dependent variable. To do so involves inverting a transcendental function. We have to resort to numerical approximations to find this parameterization. Clearly, the family of circular orbits is an exception.

3 The two-body problem

The two-body problem is to describe the motion of two bodies, of mass m_1 and m_2, under the influence of the gravitational field induced by each. Let X_1 be the vector position of m_1 and X_2 the position of m_2. Newton's law of gravitation says that the force on m_1 due to m_2 is

$$F_{12} = \frac{Gm_1 m_2}{|r|^3} r,$$

where $r = X_2 - X_1$ is the vector from m_1 to m_2, and G is the "universal gravitational constant" (see Appendix A for its value). The force on m_2 due to m_1 is $F_{21} = -F_{12}$. Newton's second law allows us to write the equations of motion for the two objects. After dividing out the common masses from both sides of the equations, we get

$$\begin{aligned}\ddot{X}_1 &= \frac{Gm_2}{|r|^3} r \,, \\ \ddot{X}_2 &= -\frac{Gm_1}{|r|^3} r \,.\end{aligned}$$

The following series of exercises illustrate how the two-body problem may be reduced to the central force problem.

Exercise 3.1. Write down the equations of motion for the relative position vector r and show that the problem of two bodies reduces to a central force problem. This means that the motion of one body relative to another traces out a conic.

Exercise 3.2. Assume the Earth's orbit around the sun is circular, and use what you've learned so far to determine the angular rotation rate and period of the Earth's orbit. How does this compare with the "experimental" value? (Assume the mass of the Earth is negligible. The radius of Earth's orbit is 1 A.U. (astronomical unit). See Appendix A).

Exercise 3.3. Towards further analysis, compute the location r_0 of the center of mass of m_1 and m_2, in terms of X_1 and X_2. The center of mass is the location on the line connecting the masses at which point the *moments* $m_1|r_0 - X_1|$ and $m_2|r_0 - X_2|$ are equal. (**Answer**: $r_0 = \frac{X_1 m_1 + X_2 m_2}{m_1 + m_2}$) Then compute the equations of motion for r_0 and show that the center of mass always moves along the linear path $r_0(t) = at + b$, for arbitrary constants a and b. Thus we can assume that the center of mass remains stationary at the origin.

Exercise 3.4. Write X_1 and X_2 in terms of r, assuming $r_0 = 0$.

4 The circular restricted three-body problem

The restricted three-body problem (RTBP) introduces a third body whose motion is affected by *but does not affect* the original two bodies, called the *primaries.* The primaries move along a conic section, as described above, and the potential well (which evolves with time) generated by the primaries controls the motion of the third body. In the circular restricted three-body problem (CRTBP) we assume a circular orbit for the primaries. This problem, especially restricted to motion of the third body in the plane of motion of the primaries, has been well studied [17].

The equations of motion

Let the center of mass of the primaries constitute the origin of our (x, y, z) coordinate system, and assume that the primaries orbit in the (x, y)-plane. Let $X(t)$ denote the position of the third body with mass m and let $X_1(t)$ and $X_2(t)$ denote the positions of the primaries.

Exercise 4.1. For the moment, assume the positions X_1 and X_2 are fixed, and use the fact that the potential (and force) due to two masses is just the sum of the individual potentials (and forces) to write down the two-body potential (and force) in terms of X_1 and X_2.

For the CRTBP we assume the primaries move in circles (of radius a for m_1 and b for m_2, say) about the origin, with common angular velocity ω.

Exercise 4.2. Use this to write down the time-dependent equations of motion for X. Assume at $t = 0$ that X_1 is on the positive x-axis, and X_2 on the negative x-axis.

Dimensionless coordinates

Through a sequence of coordinate transformations, we can reduce the number of free parameters in the equations of motion to one. The final coordinates are called *dimensionless coordinates.*

We can write the equations of motion in terms of the potential as

$$\ddot{X} = -\nabla_X V,$$

where

$$V(X,t) = -G\left(\frac{m_1}{|X - X_1|} + \frac{m_2}{|X - X_2|}\right),$$

and

$$\begin{aligned} X_1(t) &= (a\cos\omega t, a\sin\omega t, 0) \\ X_2(t) &= (-b\cos\omega t, -b\sin\omega t, 0). \end{aligned}$$

Normalize length. First, we normalize so that the distance $l = a + b$ separating the primaries is one. Thus we want to substitute $\zeta = X/l$. This means $\ddot{\zeta} = \ddot{X}/l$, so that the new equations of motion become

$$\ddot{\zeta} = -\frac{1}{l}\nabla_X V(l\zeta).$$

But now let $\tilde{\phi}(\zeta, t) = \frac{1}{l^2}V(l\zeta, t)$. Then

$$\begin{aligned} \nabla_\zeta\tilde{\phi} &= \frac{1}{l^2}\nabla_X V(l\zeta)\frac{dX}{d\zeta} \\ &= \frac{1}{l}\nabla_X V(l\zeta), \end{aligned}$$

which allows us the compact notation $\ddot{\zeta} = -\nabla\tilde{\phi}$.

Normalize time. Next we normalize the time so that the angular velocity of the primaries is one, by substituting $\tilde{t} = \omega t$. We have

$$\begin{aligned} \frac{d^2\zeta}{d\tilde{t}} &= \frac{1}{\omega^2}\frac{d^2\zeta}{dt^2} \\ &= -\frac{1}{\omega^2}\nabla\tilde{\phi}\left(\frac{\tilde{t}}{\omega}\right) \\ &= -\nabla\phi, \end{aligned}$$

where

$$\phi(\zeta,\tilde{t}) = \frac{1}{\omega^2}\tilde{\phi}(\zeta,\tilde{t}/\omega) = -\frac{G}{\omega^2 l^3}\left(\frac{m_1}{|\zeta - \frac{1}{l}X_1(\tilde{t})|} + \frac{m_2}{|\zeta - \frac{1}{l}X_2(\tilde{t})|}\right).$$

As discovered in Exercise 2.4, the angular velocity ω satisfies $\omega^2 = \frac{GM}{l^3}$, where $M = m_1 + m_2$. Thus

$$\phi(\zeta,\tilde{t}) = -\frac{1}{M}\left(\frac{m_1}{|\zeta - \frac{1}{l}X_1(\tilde{t})|} + \frac{m_2}{|\zeta - \frac{1}{l}X_2(\tilde{t})|}\right).$$

Normalize mass. Finally we normalize so that the total mass of the primaries is one. Substituting $\mu = m_1/M$ and $\mu' = m_2/M = 1-\mu$, we get

$$\phi(\zeta,\tilde{t}) = -\left(\frac{\mu}{|\zeta - \frac{1}{l}X_1(\tilde{t})|} + \frac{1-\mu}{|\zeta - \frac{1}{l}X_2(\tilde{t})|}\right).$$

And since none of the dependent variables depends on the masses, the equations of motion remain $\ddot{\zeta} = -\nabla\phi$.

Furthermore, we have the relations $\frac{b}{l} = \frac{m_1}{M}$ and $\frac{a}{l} = \frac{m_2}{M}$.

Exercise 4.3. Show this.

Thus

$$\begin{aligned}
\frac{1}{l}X_1(\tilde{t}) &= \left((1-\mu)\cos\tilde{t}, (1-\mu)\sin\tilde{t}, 0\right) \\
\frac{1}{l}X_2(\tilde{t}) &= \left(-\mu\cos\tilde{t}, -\mu\sin\tilde{t}, 0\right).
\end{aligned}$$

Set $X = (x,y,z) = \zeta$ and $\tilde{t} = t$, and define

$$\begin{aligned}
\rho_1 &= |X - X_1| \\
\rho_2 &= |X - X_2|,
\end{aligned}$$

where we redefine X_1 and X_2 to be the normalized primary vectors $(1-\mu)(\cos\tilde{t}, \sin\tilde{t}, 0)$, and $-\mu(\cos\tilde{t}, \sin\tilde{t}, 0)$, respectively. Then

$$\phi(x,y,z,t) = -\left(\frac{\mu}{\rho_1} + \frac{1-\mu}{\rho_2}\right)$$

and the final equations of motion are

$$\begin{aligned}
\ddot{x}(t) &= -\phi_x = \mu\frac{x-(1-\mu)\cos t}{\rho_1^3} + (1-\mu)\frac{x+\mu\cos t}{\rho_2^3} \\
\ddot{y}(t) &= -\phi_y = \mu\frac{y-(1-\mu)\sin t}{\rho_1^3} + (1-\mu)\frac{y+\mu\sin t}{\rho_2^3} \\
\ddot{z}(t) &= -\phi_z = \mu\frac{z}{\rho_1^3} + (1-\mu)\frac{z}{\rho_2^3}
\end{aligned}$$

The only free parameter is μ, which is usually taken to be less than or equal to $1/2$ to represent the smaller of the two primary masses.

Rotating coordinates

A disadvantage of the above representation is that the independent variable t appears explicitly in the potential function ϕ. We will now introduce another change of coordinates which takes out the rotation of the primaries, and removes time explicitly from the differential equations.

Let R_θ denote the matrix of rotation (clockwise) by angle θ about the z-axis:

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since at time t the primaries lie on a line at angle t measured counterclockwise from the x-axis, we want to make the change of coordinates $W = R_t X$.

Exercise 4.4. Show that under this change of coordinates, the equations of motion become

$$\ddot{W} - 2K\dot{W} + K^2 W = -\nabla_W U(W),$$

where

$$K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned} U(W) &= -\left(\frac{\mu}{|R_{-t}W - X_1(t)|} + \frac{1-\mu}{|R_{-t}W - X_2(t)|}\right) \\ &= -\left(\frac{\mu}{|W - (1-\mu)e_1|} + \frac{1-\mu}{|W + \mu e_1|}\right), \end{aligned}$$

and where $e_1 = (1, 0, 0)$. U is the transformed potential.

Re-introduce the variable $X = (x, y, z)$ for W, and define

$$\Omega(X) = -\frac{1}{2}\left\langle K^2 X, X\right\rangle - U(X).$$

Then the equations of motion become

$$\ddot{X} - 2K\dot{X} = \nabla_X \Omega,$$

or

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x && (1) \\ \ddot{y} + 2\dot{x} &= \Omega_y && (2) \\ \ddot{z} &= \Omega_z. && (3) \end{aligned}$$

5 Libration points

The CRTBP, when expressed by (1)-(3) in rotating coordinates, has five equilibrium points, called *Lagrangian*, or *libration* points.

Exercise 5.1. Find them! Show that equilibrium solutions occur when $\Omega_x = \Omega_y = \Omega_z = 0$. Argue that all solutions lie in the x, y-plane, and show that two of the solutions (called L_4 and L_5) form equilateral triangles with the primaries. The other three libration points (L_1–L_3) lie on the x-axis, bracketing the masses. Show that the x-coordinates of these points can be found by solving quintic polynomials.

6 The Jacobian integral

An integral of a second order system is a non-constant function $G(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$ that is constant on solutions to the system. Thus, every solution lies on some level set of G. This is expressed mathematically by the equation

$$\frac{d}{dt} G(x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t)) = 0$$

for any solution $(x_1(t), \ldots, x_n(t))$.

Exercise 6.1. Verify that

$$\mathcal{J} \equiv 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{4}$$

is an integral for the CRTBP. It is called the *Jacobian* integral, and the value it takes on a solution is called the Jacobian constant.

The Jacobian integral allows one in principle to reduce the equations of motion in the CRTBP from a 6th order system to a 5th order system. Another immediate application of the Jacobian integral is the following. A given set of initial conditions defines a particular value for the Jacobian constant, say C. The simple observation that the square of the velocity $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ must be positive implies with (4) that the range of motion in x, y, z-space for that particular solution must lie on one side of the surface defined by $\Omega(x, y, z) = C/2$. This surface is called the *surface of zero velocity* for the Jacobian constant C. (See [17, Chapter 4, Section 10.2] for properties of Ω, and a discussion of curves and surfaces of zero velocity.)

Note that despite the name, if a particle is on a surface of zero velocity, it does not necessarily have zero velocity. The surface represents a barrier through which solutions *for a particular value of the Jacobian constant* cannot pass. Solutions with different Jacobian constants will have different surfaces of zero velocity.

Exercise 6.2. Use Maple or Mathematica to graph some surfaces of zero velocity for various Jacobian constants, and identify the regions of possible motion.

7 Symmetries

Symmetries in the equations of motion allow us to find new solutions when some solutions are given. We present two symmetries here. The first symmetry is a reflection across the (x, y)-plane and the second is a reflection across the (x, z)-plane with a time-reversal. Assume that $(x(t), y(t), z(t))$ is a solution to the CRTBP, then the following are also solutions:

$$\begin{gathered}(x(t), y(t), -z(t)) \\ (x(-t), -y(-t), z(-t)) \ .\end{gathered}$$

There is also a symmetry involving parameters in the equation. This is given by:

$$(x(t), y(t), z(t), \mu) \mapsto (-x(t), -y(t), z(t), 1-\mu).$$

This means that in fact we only need to study the equations in the parameter range $0 < \mu < 1/2$. We will take advantage of the symmetry properties of the CRTBP equations in the later analysis.

Exercise 7.1. Verify these three symmetries.

8 The Lagrangian formulation

We digress slightly to present two reformulations of the equations of motion which will be of use to us later. The Lagrangian formulation will be used to develop Richardson's analytic approximation to the flow in the neighborhood of a libration point. The Hamiltonian formulation will be of use in our numerical scheme to find actual halo orbits.

The motion of a point mass in a Newtonian potential system can be formulated in terms of a Lagrangian function. The Lagrangian function is defined by

$$L(x, \dot{x}, t) = \frac{1}{2}\dot{x}^2 - U(x, t) \ ,$$

where $\dot{x}^2/2$ is the kinetic energy and $U(x, t)$ is the potential energy. The equations of motion are given by the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \ .$$

Exercise 8.1. Verify that the Lagrangian equations of motion correspond to Newton's equations of motion $\ddot{x} = -\nabla U$.

The beauty of this formulation arises from the fact that it is independent of the coordinates. The Lagrangian can be written as the difference of the kinetic and potential energies in generalized coordinates and the equations of motion may be easily derived using the Euler-Lagrange equations. This fact follows from Hamilton's principle and formulating variational equations for the action of the system. See any standard mechanics book for a detailed discussion (*e.g.*, [1, 4, 7]).

The CRTBP in dimensionless non-rotating coordinates has a Lagrangian which is given by

$$L(X, \dot{X}, t) = \frac{1}{2}\dot{X}^2 - U(X, t),$$

where

$$U(X, t) = -\frac{\mu}{|X - X_1(t)|} - \frac{1-\mu}{|X - X_2(t)|} \; .$$

We can write the Lagrangian in rotating coordinates $W = R_t X$. First calculate $X = R_{-t}W$ and $\dot{X} = \dot{R}_{-t}W + R_{-t}\dot{W}$ and substitute into the equation for L to get

$$L = \frac{1}{2}(\dot{W} - KW)^2 - U(W) \; ,$$

where

$$\begin{aligned} U(W) &= -\frac{\mu}{|W - T_t X_1(t)|} - \frac{1-\mu}{|W - R_t X_2(t)|} \\ &= -\frac{\mu}{|W - (1-\mu)e_1|} - \frac{1-\mu}{|W + \mu e_1|} \; . \end{aligned}$$

Exercise 8.2. Verify that the equations of motion in rotating coordinates derived from using this Lagrangian and the Euler-Lagrange equations are the same as the ones we derived earlier.

9 The Hamiltonian formulation

The CRTBP is a *Hamiltonian* dynamical system. This means that there exist coordinates p, q and a function $H(p, q, t)$ such that the equations of motion are given by

$$\begin{aligned} \dot{q} &= \partial H/\partial p \\ \dot{p} &= -\partial H/\partial q \; . \end{aligned} \tag{5}$$

This is often written in the compact form

$$\dot{z} = J\nabla_z H(z) \; ,$$

where $z = (q, p)$ and J is the matrix of four square blocks given by

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \; .$$

For a discussion of this extensive theory, see, for example, [1, 4, 7, 10].

We will write down the Hamiltonian and corresponding canonical coordinates in rotating variables since this will be most useful for us. Typically this is done by starting from the Lagrangian written using the traditional coordinates $q, \dot{q}$. Thus, we start with

$$L(q, \dot{q}, t) = \frac{1}{2}(\dot{q} - Kq)^2 - U(q) \; .$$

The generalized momenta p are given by $p = \partial L/\partial \dot{q}$, so

$$p = \dot{q} - Kq \ .$$

The Hamiltonian is defined by $H(p, q, t) =< p, \dot{q} > -L$, so we compute

$$H = \frac{1}{2}p^2 + < p, Kq > +U(q) \ .$$

The equations of motion are given by applying Equation 5 to this Hamiltonian.

Exercise 9.1. Verify that these equations of motion coincide with those from the Lagrangian formulation.

Since the Hamiltonian for this problem is independent of time, it is a constant of the motion. This can be verified by computing $dH/dt = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = 0$.

Exercise 9.2. Verify that $H = -\mathcal{J}/2$, where $\mathcal{J}$ is the Jacobian integral introduced in Section 6.

10 Richardson's approximations for halo orbits

In this section we discuss a third-order analytic approximation to the equations of motion which produces periodic solutions about the collinear libration points. Our discussion follows the work of D. L. Richardson [14, 15, 16], who used this technique successfully in designing orbits for the ISEE3 mission to L_1 in the late 1970s.

Yet another change of coordinates

We will discuss flow in a neighborhood of L_1 and L_2, recalling that flow near L_3 is the same as that near L_2 but for a different parameter value. In order to study the motion near a libration point it is easiest to choose a coordinate system that is centered at the libration point. We will also normalize distances so that the distance between the libration point and the mass M_1 is one unit.

In the rotating coordinate system, let the locations of the libration points L_i be given by

$$L_i = \alpha_i e_1 \ ,$$

where the α_i are roots of the appropriate quintic as discussed in an earlier section. Let the vectors from L_i to the masses M_1 and M_2 be given by

$$\begin{aligned} r_1 &= (1-\mu)e_1 - L_i = (1-\mu-\alpha_i)e_1 \ , \\ r_2 &= -\mu e_1 - L_i = -(\mu+\alpha_i)e_1 \ . \end{aligned}$$

New rotating coordinates centered at the libration point L_i and rescaled by $|r_1|$ are given by

$$\rho = (W - L_i)/|r_1| \ .$$

We write the Lagrangian in these new coordinates as

$$\begin{aligned} L(\rho,\dot\rho) &= \frac{|r_1|^2}{2}(\dot\rho - K\rho)^2 + \alpha_i|r_1| < \rho, e_1 > -U(\rho) \\ &\quad +\alpha_i|r_1| < \dot\rho, e_2 > +\frac{1}{2}\alpha_i^2 \ , \end{aligned}$$

with

$$U(\rho) = \frac{-\mu/|r_1|}{|\rho \mp e_1|} + \frac{-(1-\mu)/|r_1|}{|\rho + \frac{|r_2|}{|r_1|}e_1|} \ ,$$

where from now onwards the top sign is taken for L_1 and the bottom for L_2. We want to develop this as a power series in $|\rho|$ and thus want $|r_2| \geq |r_1|$. Consequently, this analysis will give results for L_1 when $0 < \mu \leq 1/2$ and for L_2 when $0 < \mu < 1$. From the symmetry discussed earlier we can produce orbits for L_1 when $1/2 < \mu < 1$ by applying the analysis for L_1 using $1-\mu$. Similarly, we can study orbits around L_3 by using the symmetry applied to orbits about L_2.

Note that rescaling of the Lagrangian by a constant does not affect the equations of motion so we shall divide by $|r_1|^2$. Also, constants and linear terms which contain only $\dot\rho$ play no role in the equations of motion, so the final two terms may be discarded. We now write the Lagrangian as

$$L(\rho,\dot\rho) = \frac{1}{2}(\dot\rho - K\rho)^2 + \alpha_i\frac{\rho_x}{|r_1|} - U(\rho) \ ,$$

with

$$U(\rho) = \frac{-\mu}{|r_1|^3}\cdot\frac{1}{|\rho \mp e_1|} + \frac{-(1-\mu)}{|r_1|^2|r_2|}\cdot\frac{|r_2|/|r_1|}{|\rho + \frac{|r_2|}{|r_1|}e_1|} \ .$$

The next step is to write the potential function U using a power series of Legendre polynomials.

To introduce Legendre polynomials we recall a couple of identities. First an application of the law of cosines gives

$$\frac{|r|}{|\rho - r|} = \left[1 - 2\frac{|\rho|}{|r|}\cos\theta + \frac{|\rho|^2}{|r|^2}\right]^{-1/2} \ ,$$

where θ is the angle between ρ and r. Then we recognize that $(1-2zx+z^2)^{-1/2}$ is the generating function for the Legendre polynomials. In other words,

$$(1 - 2z\cos\theta + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos\theta) z^n \ ,$$

and

$$\frac{|r|}{|\rho - r|} = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{|\rho|^n}{|r|^n} \ .$$

The Legendre polynomials can also be developed from a recursion relation given by

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_1(\cos\theta) &= \cos\theta \\ P_n(\cos\theta) &= \left(\frac{2n-1}{n}\cos\theta\right)P_{n-1} - \left(\frac{n-1}{n}\right)P_{n-2} \ . \end{aligned}$$

Note that P_i is even or odd in $\cos\theta$ exactly when i is even or odd. Also recall the formula for the derivative of a Legendre polynomial,

$$\frac{dP_n(C)}{dC} = \sum_{0\le k\le (n-1)/2, k\in\mathbb{Z}} (2n-4k-1)P_{n-2k-1}(C) \ .$$

Applying this to terms in our potential function we see that

$$\begin{aligned}
\frac{1}{|\rho \mp e_1|} &= \sum_{n=0}^{\infty} P_n(\pm\rho_x/|\rho|)|\rho|^n \ , \\
\frac{|r_2|/|r_1|}{|\rho + \frac{|r_2|}{|r_1|}e_1|} &= \sum_{n=0}^{\infty} P_n(-\rho_x/|\rho|)\frac{|r_1|^n}{|r_2|^n}|\rho|^n \ .
\end{aligned}$$

When introducing the expansions into the Lagrangian we can once again neglect the constant terms and we will collect the linear terms. This results in a Lagrangrian of

$$\begin{aligned}
L(\rho,\dot\rho) &= \frac{1}{2}(\dot\rho - K\rho)^2 + \frac{\mu}{|r_1|^3}\sum_{n=2}^{\infty} P_n(\pm\rho_x/|\rho|)|\rho|^n \\
&\quad + \frac{(1-\mu)}{|r_1|^2|r_2|}\sum_{n=2}^{\infty} P_n(-\rho_x/|\rho|)\frac{|r_1|^n}{|r_2|^n}|\rho|^n \\
&\quad + \frac{\rho_x}{|r_1|}\left(\alpha_i \pm \frac{\mu}{|r_1|^2} - \frac{1-\mu}{|r_2|^2}\right) \ .
\end{aligned}$$

Now recall that

$$\alpha_i \pm \frac{\mu}{|r_1|^2} - \frac{1-\mu}{|r_2|^2} = \alpha_i \pm \frac{\mu}{(\alpha_i - (1-\mu))^2} - \frac{1-\mu}{(\alpha_i+\mu)^2} = 0 \ ,$$

since this is the equation which defines the libration point! Of course we know this quantity will be zero simply because we have placed the origin of the coordinate system at an equilibrium point thus guaranteeing that linear terms will not exist.

Combining the sums in the Lagrangian, we write

$$L(\rho,\dot\rho) = \frac{1}{2}(\dot\rho - K\rho)^2 + \sum_{n=2}^{\infty} c_n P_n(\rho_x/|\rho|)|\rho|^n \ ,$$

where

$$\begin{aligned}
c_n &= (\pm 1)^n\frac{\mu}{|r_1|^3} + (-1)^n\frac{(1-\mu)|r_1|^{n-2}}{|r_2|^{n+1}} \\
&= (\pm 1)^n\frac{\mu}{|\alpha_i - (1-\mu)|^3} + (-1)^n(1-\mu)\frac{|\alpha_i - (1-\mu)|^{n-2}}{|\alpha_i+\mu|^{n+1}} \ .
\end{aligned}$$

If you insist on finding the equations of motion in a neighborhood of L_3, then in the coordinates $\rho = (L_3 - W)/|r_2|$ the above Lagrangian holds and the c_n are the same as those for L_2 except they must be computed for parameter value $1-\mu$. Performing this computation, we find that the c_n for the Lagrangian at L_3 are given by

$$c_n = (-1)^n \frac{1-\mu}{|r_2|^3} + (-1)^n \frac{\mu |r_2|^{n-2}}{|r_1|^{n+1}} \; .$$

The equations of motion are derived from the Lagrangian using the Euler-Lagrange equations. Consequently, the equations of motion are

$$\begin{aligned} \ddot{\rho} - 2K\dot{\rho} + K^2\rho &= \sum_{n=2}^{\infty} n c_n P_n(\rho_x/|\rho|)|\rho|^{n-2}\rho \\ &\quad + \sum_{n=2}^{\infty} c_n P_n'(\rho_x/|\rho|)|\rho|^{n-2}(|\rho|e_1 - \frac{\rho_x}{|\rho|}\rho) \; . \end{aligned}$$

We can now approximate the dynamical system near the libration points by using perturbation methods. Essentially, we will expand all quantities in a power series of a small parameter (*e.g.*, the amplitude of the solution) and consider only a finite number of terms in the infinite sum. We can then attempt to find explicit solutions for these dynamical systems and hope that they approximate solutions of the full problem.

The linear approximation

First we will study a linear approximation to the equations of motion just developed. We show that solutions to such an approximation are inadequate in the sense that they are not, in general, periodic. We will require these solutions, however, when we move to higher-order approximations.

The linear approximation involves taking only the quadratic terms in the Lagrangian. This gives us

$$L_2(\rho, \dot{\rho}) = \frac{1}{2}(\dot{\rho} - K\rho)^2 + c_2(3\rho_x^2 - \rho^2)/2$$

with

$$c_2 = \frac{\mu}{|\alpha_i - (1-\mu)|^3} + \frac{1-\mu}{|\alpha_i + \mu|^3} \; .$$

Using the Euler-Lagrange equations we write down the equations of motion to get

$$\ddot{\rho} - 2K\dot{\rho} + \begin{pmatrix} -(2c_2+1) & 0 & 0 \\ 0 & c_2 - 1 & 0 \\ 0 & 0 & c_2 \end{pmatrix} \rho = 0 \; .$$

This is a linear degree two system of ordinary differential equations which we can solve by spectral methods. Note that the z-component is decoupled from the other two. We compute the characteristic polynomial to be

$$(\Lambda^2 + c_2)(\Lambda^4 + \Lambda^2(2-c_2) + 1 + c_2 - 2c_2^2) = 0,$$

and solving for Λ find

$$\Lambda = \pm j\sqrt{c_2}, \Lambda = \pm\sqrt{\frac{c_2 - 2 \pm \sqrt{9c_2^2 - 8c_2}}{2}}$$

where j represents the complex involute. If $c_2 > 1$ then $9c_2^2 - 8c_2 > (c_2 - 2)^2$ and there will be two pairs of purely imaginary eigenvalues and one pair of real eigenvalues.

Exercise 10.1. Show that $c_2 > 1$.

We can thus write the solution to the linear equation in the form:

$$\begin{aligned} x(t) &= A_1 e^{\alpha t} + A_2 e^{-\alpha t} + A_3 \cos \lambda t + A_4 \sin \lambda t \\ y(t) &= -k_1 A_1 e^{\alpha t} + k_1 A_2 e^{-\alpha t} - k_2 A_3 \sin \lambda t + k_2 A_4 \cos \lambda t \\ z(t) &= A_5 \cos \sqrt{c_2} t + A_6 \sin \sqrt{c_2} t \ , \end{aligned}$$

where

$$\begin{aligned} \alpha &= \sqrt{\frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}} \\ \lambda &= \sqrt{-\frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}} \\ k_1 &= (2c_2 + 1 - \alpha^2)/2\alpha \\ k_2 &= (2c_2 + 1 + \lambda^2)/2\lambda \ , \end{aligned}$$

and $A_1, \ldots, A_6$ are arbitrary constants determined by the initial conditions.

Since we are interested in periodic and quasiperiodic solutions we take $A_1 = A_2 = 0$. Then the solution to the linear problem can be written in terms of amplitude and phase as

$$\begin{aligned} x(t) &= -A_x \cos(\lambda t + \phi) \\ y(t) &= k A_x \sin(\lambda t + \phi) \\ z(t) &= A_z \sin(\sqrt{c_2} t + \psi) \ , \end{aligned}$$

with $k = k_2$. For out of plane solutions ($A_z \neq 0$), we do not expect λ and c_2 to be rationally related and thus expect quasiperiodic solutions. We must therefore include nonlinearities if we hope to find periodic solutions.

The method of Lindstedt-Poincaré

To find better approximations to the nonlinear problem in a neighborhood of the equilibrium point solutions we will use the perturbation techniques of Lindstedt-Poincaré [11].

The first thing we do is allow for a frequency correction by setting $\tau = \omega t$ and letting $''$' denote $d/d\tau$. We will then truncate the equations of motion at degree 3. Doing this we get

$$\begin{aligned} \omega^2 x'' - 2\omega y' - (1 + 2c_2)x &= \frac{3}{2} c_3 (2x^2 - y^2 - z^2) \\ &\quad + 2c_4 (2x^2 - 3y^2 - 3z^2)x \\ \omega^2 y'' + 2\omega x' + (c_2 - 1)y &= -3c_3 xy - \frac{3}{2} c_4 (4x^2 - y^2 - z^2)y \\ \omega^2 z'' + \lambda^2 z &= -3c_3 xz - \frac{3}{2} c_4 (4x^2 - y^2 - z^2)z + \Delta z \ , \end{aligned}$$

where

$$\Delta = \lambda^2 - c_2 \ .$$

We continue the perturbation analysis by assuming solutions of the form:

$$\begin{aligned}
x(\tau) &= \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \epsilon^3 x_3(\tau) + \dots \\
y(\tau) &= \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \epsilon^3 y_3(\tau) + \dots \\
z(\tau) &= \delta z_1(\tau) + \delta^2 z_2(\tau) + \delta^3 z_3(\tau) + \dots
\end{aligned}$$

and letting

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \ .$$

We substitute these quantities into the equations of motion and equate components of the same order. Here we make the assumption that $\Delta = O(\epsilon^2)$ and set $\delta = \epsilon$.

The first order equations

The $O(\epsilon)$ equations are the linearization of the vector field which we solved in the previous section, with a modified frequency for the out of plane oscillations:

$$\begin{aligned}
x_1'' - 2y_1' - (1 + 2c_2)x_1 &= 0 \\
y_1'' + 2x_1' + (c_2 - 1)y_1 &= 0 \\
z_1'' + \lambda^2 z_1 &= 0 \ .
\end{aligned}$$

Since we are only interested in bounded solutions we write the solutions

$$\begin{aligned}
x_1(\tau) &= -A_x \cos(\lambda\tau + \phi) \\
y_1(\tau) &= kA_x \sin(\lambda\tau + \phi) \\
z_1(\tau) &= A_z \sin(\lambda\tau + \psi) \ .
\end{aligned}$$

Later, in order to avoid secular solutions we will need to put constraints on the constants A_x, A_z, ϕ, ψ, but for now they are arbitrary.

The second order equations

The $O(\epsilon^2)$ equations depend on the above solutions and are given by

$$\begin{aligned}
x_2'' - 2y_2' - (1 + 2c_2)x_2 &= -2\omega_1(x_1'' - y_1') + \frac{3}{2}c_3(2x_1^2 - y_1^2 - z_1^2) \\
y_2'' + 2x_2' + (c_2 - 1)y_2 &= -2\omega_1(y_1'' + x_1') - 3c_3x_1y_1 \\
z_2'' + \lambda^2 z_2 &= -2\omega_1 z_1'' - 3c_3x_1z_1 \ .
\end{aligned}$$

Substituting in the solutions for x_1, y_1, z_1, we get

$$
\begin{aligned}
x_2'' - 2y_2' - (1+2c_2)x_2 &= +2\omega_1 \lambda A_x (k-\lambda)\cos\tau_1 \\
&\quad +\alpha_1 + \gamma_1 \cos 2\tau_1 + \gamma_2 \cos 2\tau_2 \\
y_2'' + 2x_2' + (c_2-1)y_2 &= 2\omega_1 A_x \lambda (k\lambda - 1)\sin\tau_1 \\
&\quad +\beta_1 \sin 2\tau_1 \\
z_2'' + \lambda^2 z_2 &= 2\omega_1 A_z \lambda^2 \sin\tau_2 \\
&\quad +\delta_1 \sin(\tau_1+\tau_2) + \delta_1 \sin(\tau_2 - \tau_1)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_1 &= \lambda\tau + \phi \\
\tau_2 &= \lambda\tau + \psi\ ,
\end{aligned}
$$

and the expressions for the other coefficients are given in Appendix C. These are a set of inhomogeneous linear differential equations whose solutions are summarized in Appendix B. We know the bounded homogeneous solution which is incorporated into the solution to the first order equations and simply need to find a particular solution. The secular terms (those responsible for producing unbounded solutions) are the $\sin\tau_1$, $\cos\tau_1$ and $\sin\tau_2$ terms which are all eliminated by setting $\omega_1 = 0$. Thus we find the solution

$$
\begin{aligned}
x_2 &= \rho_{20} + \rho_{21}\cos 2\tau_1 + \rho_{22}\cos 2\tau_2 \\
y_2 &= \sigma_{21}\sin 2\tau_1 + \sigma_{22}\sin 2\tau_2 \\
z_2 &= \kappa_{21}\sin(\tau_1+\tau_2) + \kappa_{22}\sin(\tau_2-\tau_1)\ ,
\end{aligned}
$$

where the expressions for the coefficients are given in Appendix C.

The third order equations

The $O(\epsilon^3)$ equations are (after setting $\omega_1 = 0$)

$$
\begin{aligned}
x_3'' - 2y_3' - (1+2c_2)x_3 &= -2\omega_2(x_1'' - y_1') \\
&\quad +3c_3(2x_1x_2 - y_1y_2 - z_1z_2) \\
&\quad +2c_4x_1(2x_1^2 - 3y_1^2 - 3z_1^2) \\
y_3'' + 2x_3' + (c_2-1)y_3 &= -2w_2(x_1' + y_1'') \\
&\quad -3c_3(x_1y_2 + x_2y_1) \\
&\quad -\frac{3}{2}c_4y_1(4x_1^2 - y_1^2 - z_1^2) \\
z_3'' + \lambda^2 z_3 &= -2\omega_2 z_1'' + \frac{\Delta}{\epsilon^2}z_1 \\
&\quad -3c_3(x_2z_1 + x_1z_2) \\
&\quad -\frac{3}{2}c_4z_1(4x_1^2 - y_1^2 - z_1^2)\ .
\end{aligned}
$$

Substituting in the solutions for $x_1, y_1, z_1, x_2, y_2, z_2$, we get

$$\begin{aligned}
x_3'' - 2y_3' - (1+2c_2)x_3 &= [\nu_1 + 2\omega_2\lambda A_x(k-\lambda)]\cos\tau_1 \\
&\quad +\gamma_3\cos 3\tau_1 + \gamma_4\cos(\tau_1+2\tau_2) \\
&\quad +\gamma_5\cos(2\tau_2-\tau_1) \\
y_3'' + 2x_3' + (c_2-1)y_3 &= [\nu_2 + 2\omega_2\lambda A_x(\lambda k-1)]\sin\tau_1 \\
&\quad +\beta_3\sin 3\tau_1 + \beta_4\sin(\tau_1+2\tau_2) \\
&\quad +\beta_5\sin(2\tau_2-\tau_1) \\
z_3'' + \lambda^2 z_3 &= [\nu_3 + A_z(2\omega_2\lambda^2 + \Delta/\epsilon^2)]\sin\tau_2 \\
&\quad +\delta_3\sin 3\tau_2 + \delta_4\sin(2\tau_1+\tau_2) \\
&\quad +\delta_5\sin(2\tau_1-\tau_2)\ ,
\end{aligned}$$

where the expressions for the coefficients are given in Appendix C. There are secular terms in the $x_3 - y_3$ equations and in the z_3 equations which can no longer be removed by simply setting a value for the frequency correction ω_2. We start by examining the secular terms in the z_3 equations more closely.

To remove the $\delta_5\sin(2\tau_1 - \tau_2)$ term we would need $\delta_5 = 0$. Since we cannot freely adjust this parameter, we can attempt to remove the secular term by adjusting the phases of τ_1 and τ_2 so that $\sin(2\tau_1 - \tau_2) \sim \sin\tau_2$. To do this we need

$$\phi = \psi + n\frac{\pi}{2}\ ,$$

where $n = 0, 1, 2, 3$. Then the z_3 solution will be bounded if

$$\nu_3 + A_z(2\omega_2\lambda^2 + \Delta/\epsilon^2) + \zeta\delta_5 = 0\ ,$$

where $\zeta = (-1)^n$. This phase constraint affects the $x_3 - y_3$ equations, which now each contain one secular term. Instead of forcing the removal of these terms with two additional constraint equations, we see from Appendix C that we can simultaneously eliminate the unbounded terms from both particular solutions with a single constraint equation. The requirement is

$$(\nu_1 + 2\omega_2\lambda A_x(k-\lambda) + \zeta\gamma_5) - k(\nu_2 + 2\omega_2\lambda A_x(\lambda k - 1) + \zeta\beta_5) = 0\ .$$

We satisfy this last equation by setting

$$\begin{aligned}
\omega_2 &= \frac{\nu_1 - k\nu_2 + \zeta(\gamma_5 - k\beta_5)}{2\lambda A_x[\lambda(1+k^2) - 2k]} \\
&= s_1 A_x^2 + s_2 A_z^2\ ,
\end{aligned}$$

where the expressions for s_1, s_2 are given in Appendix C. Substituting this into the first constraint we get

$$l_1 A_x^2 + l_2 A_z^2 + \Delta/\epsilon^2 = 0\ ,$$

with l_1, l_2 given in Appendix C, and thus can satisfy this constraint by letting one amplitude be determined by the other.

Assuming these constraints the third order equations reduce to

$$
\begin{aligned}
x_3'' - 2y_3' - (1+2c_2)x_3 &= k\beta_6 \cos\tau_1 + (\gamma_3 + \zeta\gamma_4)\cos 3\tau_1 \\
y_3'' + 2x_3' + (c_2 - 1)y_3 &= \beta_6 \sin\tau_1 + (\beta_3 + \zeta\beta_4)\sin 3\tau_1 \\
z_3'' + \lambda^2 z_3 &= \begin{cases} (-1)^{n/2}(\delta_3 + \delta_4)\sin 3\tau_1 & n = 0,2 \\ (-1)^{(n-1)/2}(\delta_4 - \delta_3)\cos 3\tau_1 & n = 1,3 \end{cases} ,
\end{aligned}
$$

where $\beta_6 = \nu_2 + 2\omega_2\lambda A_x(\lambda k - 1) + \zeta\beta_5$. The solution to this is given by

$$
\begin{aligned}
x_3 &= \rho_{31}\cos 3\tau_1 \\
y_3 &= \sigma_{31}\sin 3\tau_1 + \sigma_{32}\sin\tau_1 \\
z_3 &= \begin{cases} (-1)^{n/2}\kappa_{31}\sin 3\tau_1 & n = 0,2 \\ (-1)^{(n-1)/2}\kappa_{32}\cos 3\tau_1 & n = 1,3 \end{cases} ,
\end{aligned}
$$

where the expressions for the coefficients are given in Appendix C. Richardson's solution [14, 15, 16] neglected the β_6 terms and thus did not include the σ_{32} correction – a cubic correction to the amplitude of the primary frequency.

The final approximation

Applying the mapping $A_x \mapsto A_x/\epsilon$ and $A_z \mapsto A_z/\epsilon$ will remove ϵ from all the equations. Then we can use A_x or A_z as a small parameter. Combining the solutions we have computed to third order, we get

$$
\begin{aligned}
x(\tau) &= \rho_{20} - A_x\cos\tau_1 + (\rho_{21} + \zeta\rho_{22})\cos 2\tau_1 + \rho_{31}\cos 3\tau_1 \\
y(\tau) &= (kA_x + \sigma_{32})\sin\tau_1 + (\sigma_{21} + \zeta\sigma_{22})\sin 2\tau_1 + \sigma_{31}\sin 3\tau_1 \\
z(\tau) &= \begin{cases} (-1)^{n/2}A_z\sin\tau_1 & n = 0,2 \\ (-1)^{(n-1)/2}A_z\cos\tau_1 & n = 1,3 \end{cases} \\
&\quad + \begin{cases} (-1)^{n/2}(\kappa_{21}\sin 2\tau_1 + \kappa_{31}\sin 3\tau_1) & n = 0,2 \\ (-1)^{(n-1)/2}(\kappa_{21}\cos 2\tau_1 + \kappa_{22} + \kappa_{32}\cos 3\tau_1) & n = 1,3 \end{cases} .
\end{aligned}
$$

These equations may be rewritten to more clearly see the dependence on the small parameters A_x and A_z. This is

$$
\begin{aligned}
x(\tau) &= -A_x \cos\tau_1 + a_{21}A_x^2 + a_{22}A_z^2 + (a_{23}A_x^2 + \zeta a_{24}A_z^2)\cos 2\tau_1 \qquad (6)\\
&\quad +(a_{31}A_x^3 + \zeta a_{32}A_xA_z^2)\cos 3\tau_1\\
y(\tau) &= kA_x \sin\tau_1 + (b_{21}A_x^2 + \zeta b_{22}A_z^2)\sin 2\tau_1\\
&\quad +(b_{31}A_x^3 + \zeta b_{32}A_xA_z^2)\sin 3\tau_1 \qquad (7)\\
&\quad +(b_{33}A_x^3 + b_{34}A_xA_z^2 + \zeta b_{35}A_xA_z^2)\sin\tau_1\\
z(\tau) &= \begin{cases} (-1)^{n/2}A_z \sin\tau_1 & n=0,2\\ (-1)^{(n-1)/2}A_z\cos\tau_1 & n=1,3\end{cases}\\
&\quad + \begin{cases} (-1)^{n/2}d_{21}A_xA_z\sin 2\tau_1 & n=0,2\\ (-1)^{(n-1)/2}d_{21}A_xA_z(\cos 2\tau_1 - 3) & n=1,3\end{cases} \qquad (8)\\
&\quad + \begin{cases} (-1)^{n/2}(d_{31}A_z^3 + d_{32}A_x^2A_z)\sin 3\tau_1 & n=0,2\\ (-1)^{(n-1)/2}(d_{32}A_zA_x^2 - d_{31}A_z^3)\cos 3\tau_1 & n=1,3\ .\end{cases}
\end{aligned}
$$

The expressions for the coefficients are given in Appendix C.

11 Numerical computation of halo orbits

In this section we discuss some strategies for the numerical computation of periodic orbits. The method we will discuss is basically a Newton's method scheme, with the idea that Richardson's analytic approximation will provide convergent initial seeds for the method. We will first examine the problem of computing periodic orbits in the 2-degrees of freedom CRTBP – that is, orbits which lie in the (x, y)-coordinate plane. Then we will look at out-of-plane orbits. In both cases we will restrict the discussion to finding orbits which have a symmetry across the x, z-plane.

Symmetric periodic orbits for the 2-degrees of freedom CRTBP

We would like to find a periodic orbit which lies in the x, y-coordinate plane and is symmetric about the x-axis. There are families of such orbits surrounding the collinear libration points. We shall define one of these orbits by the location of its intersection with the x-axis and its velocity. Thus if a point in phase space is given by $X = (x, y, v_x, v_y)$, then a periodic orbit can be specified by $(x, 0, 0, v_y)$. We have set $y = 0$ to specify the intersection of the orbit with the x-axis, and we set $v_x = 0$ in order to satisfy the symmetry condition.

Suppose that the orbit only crosses the x-axis in two points. Then the second point will be halfway around. We can then use the symmetry of the problem to generate the entire orbit from the half of the orbit lying to one side of the x-axis.

Let $\dot{X} = f(X)$ be the equations of motion. We are of course here only interested in the CRTBP, but these techniques will work in a great variety of problems. Let $\Phi(X, t)$ be the solution to the differential equation, which means that $\Phi(X, 0) = X$ and $\partial\Phi(X, t)/\partial t = f(\Phi(X, t))$.

Let $X = (x, 0, 0, v_y)$ be an initial guess for a point on a symmetric planar periodic orbit of the CRTBP. Flow the point under the vector field until $y = 0$ again and let $T_{1/2}$ be the time of the flow. Thus $\Phi(X, T_{1/2}) = (\tilde{x}, 0, \tilde{v}_x, \tilde{v}_y)$. If $\tilde{v}_x = 0$ then the initial guess is part of the periodic orbit and we are done; if not then we need to improve the initial guess in order to attempt to drive $\tilde{v}_x$ to zero. Thus we compute

$$\begin{aligned} \Phi(X + \Delta X, T_{1/2} + \Delta t) &= \Phi(X, T_{1/2}) + \left[\frac{\partial \Phi(X, T_{1/2})}{\partial X}\right] \cdot \Delta X \\ &\quad + \frac{\partial \Phi(X, T_{1/2})}{\partial t} \cdot \Delta t + \text{h.o.t.}\,. \end{aligned}$$

Since we wish to restrict our choice of initial conditions, we let $\Delta X = (\Delta x, 0, 0, \Delta v_y)$ and solve the following equations which come from the approximation found by dropping the higher order terms in the above equation:

$$\left[\frac{\partial \Phi}{\partial X}\right] \begin{pmatrix} \Delta x \\ 0 \\ 0 \\ \Delta v_y \end{pmatrix} + f(\Phi(X, T_{1/2})) \cdot \Delta t = \begin{pmatrix} * \\ 0 \\ -\tilde{v}_x \\ * \end{pmatrix} .$$

We use '$*$' in the above equation to indicate quantities whose values do not concern us. So we end up with exactly two equations and three unknowns. This is expected since we will find a family of periodic solutions which can be parameterized by, for example, the intersection with the x-axis. Fixing this quantity, we must take $\Delta x = 0$ and then can solve for Δv_y and Δt. This will give us a new guess for the initial velocity $v_y + \Delta v_y$ and an estimate of $T_{1/2} + \Delta t$ for the return time. The whole process may be repeated.

We have not yet discussed the quantity $\left[\frac{\partial \Phi}{\partial X}\right]$. Since we do not have an explicit formula for $\Phi(X, t)$, we cannot compute this matrix of functions explicitly. In the next section we discuss this quantity, which is known as the fundamental solution matrix.

The fundamental solution matrix

Let $\Phi(X, t)$ solve the n-dimensional differential equation $\dot{X} = f(X)$, so $\Phi(X, 0) = X$ and $\partial \Phi(X, t)/\partial t = f(\Phi(X, t))$. If we wish to see how a small change in initial condition X will affect the solution after time t, then to approximate this to first order we must compute $\left[\frac{\partial \Phi}{\partial X}\right]$. This matrix of functions is a particular fundamental solution matrix. The role of this matrix should be clear from the following calculation:

$$\begin{aligned} \Phi(X + \Delta X, t + \Delta t) &= \Phi(X, t) + \left[\frac{\partial \Phi(X, t)}{\partial X}\right] \cdot \Delta X + \frac{\partial \Phi(X, t)}{\partial t} \cdot \Delta t \\ &\quad + \text{h.o.t.} \end{aligned}$$

Since $\Phi(X, 0) = X$, we compute

$$\left[\frac{\partial \Phi(X, 0)}{\partial X}\right] = \mathbf{1}\,.$$

Thus for $t = 0$ the matrix is the identity matrix. We can then see how the matrix evolves in time by computing

$$\frac{\partial}{\partial t}\left[\frac{\partial \Phi}{\partial X}\right] = \frac{\partial}{\partial X}\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial X} f(\Phi) = \left[\frac{\partial f}{\partial X}\right]|_{X=\Phi}\left[\frac{\partial \Phi}{\partial X}\right] .$$

Thus we have a linear, nonautonomous differential equation in n^2 dimensions which describes the evolution of a fundamental solution matrix. These are sometimes called the first variational equations. We can set up a system to simultaneously compute the solution Φ and the fundamental solution matrix $\left[\frac{\partial \Phi}{\partial X}\right]$, by augmenting the initial n dimensional vector field by n^2 dimensions. Letting M denote the fundamental solution matrix, the resulting $n^2 + n$ dimensional initial value problem that we solve is:

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{M} &= \left[\frac{\partial f}{\partial X}\right] M , \end{aligned}$$

with initial conditions

$$\begin{aligned} X(0) &= X_0 \\ M(0) &= \mathbf{1} . \end{aligned}$$

When the solution is a periodic orbit and this fundamental solution matrix is computed at a time equal to one period, then the resulting matrix is called the monodromy matrix. The eigenvalues of this matrix determine the stability of the periodic orbit.

Halo orbits - symmetric periodic orbits for the 3-degrees of freedom CRTBP

In this section we extend the previous discussion regarding the computation of periodic orbits to the full 3 degrees of freedom problem. This is nearly identical to the previous discussion, except that the dimensions are increased. These orbits surrounding the libration points L_1, L_2, and L_3 are called halo orbits.

Let $X = (x, y, z, v_x, v_y, v_z)$. An initial guess is now located on the (x, z)-plane with a component of velocity only in the y direction. Let $X = (x, 0, z, 0, v_y, 0)$ be an initial guess. If $T_{1/2}$ is the time of first return to the (x, z)-plane, then we compute $\Phi(X, T_{1/2}) = (\tilde{x}, 0, \tilde{z}, \tilde{v}_x, \tilde{v}_y, \tilde{v}_z)$. We are looking for initial conditions that give us $\tilde{v}_x = \tilde{v}_z = 0$ in order to give us the first half of a periodic orbit. Adjusting the initial condition and flow time we get:

$$\begin{aligned} \Phi(X + \Delta X, T_{1/2} + \Delta t) &= \Phi(X, T_{1/2}) + \left[\frac{\partial \Phi(X, T_{1/2})}{\partial X}\right] \cdot \Delta X \\ &\quad + \frac{\partial \Phi(X, T_{1/2})}{\partial T} \cdot \Delta t + \text{h.o.t.} . \end{aligned}$$

Restricting ΔX to $\Delta X = (\Delta x, 0, \Delta z, 0, \Delta v_y, 0)$, we find a new guess by solving:

$$\left[\frac{\partial \Phi}{\partial X}\right] \begin{pmatrix} \Delta x \\ 0 \\ \Delta z \\ 0 \\ \Delta v_y \\ 0 \end{pmatrix} + f(\Phi(X, T_{1/2}) \cdot \Delta t = \begin{pmatrix} * \\ 0 \\ * \\ -\tilde{v}_x \\ * \\ -\tilde{v}_z \end{pmatrix} .$$

The '$*$' represent quantities we do not care about, so we end up with three equations and four unknowns. Thus we can parameterize the family of solutions by one quantity, say the out of plane amplitude z, in which case we require that $\Delta z = 0$. This allows us to solve for a new initial x-position $x + \Delta x$ and new initial y-velocity $v_y + \Delta v_y$. This process is repeated in the expectation of having found a better approximation to a periodic orbit. In Section 14 we present the results of applying this algorithm.

Another type of symmetric periodic orbit is suggested by cases $n = 0, 2$ in the Richardson approximation (6) – (8). These orbits cross the x-axis perpendicularly at $\tau_1 = 0, \pi$. Any solution to the CRTBP with this characteristic would necessarily be symmetric across the x-axis. Nevertheless, we have been unable to apply techniques similar to those above to locate solutions with this symmetry close to the $n = 0, 2$ Richardson approximation.

12 Stability of the periodic orbits

In order to compute the stability of a periodic orbit, one must compute the eigenvalues of the monodromy matrix. Recall that the monodromy matrix is a particular fundamental solution matrix for the variational equations along a periodic orbit evaluated after one period of the orbit. Although the monodromy matrix is defined in terms of a specific point on the periodic orbit, all the monodromy matrices (for different points on the orbit) are related by similarity transformations. Thus the eigenvalues of the monodromy matrix are an invariant quantity. The monodromy matrix M from a volume preserving Hamiltonian system is a symplectic mapping — that is, it satisfies $M^{-1} = -JM^*J$, where * denotes matrix transpose and J is the matrix introduced in Section 9 (see [10], for example). Symplectic matrices have the following properties:

1. $\det(M) = 1$.

2. If λ is an eigenvalue of M, then so is $1/\lambda$.

In practice the monodromy matrix is computed by numerical integration. When the periodic orbit is symmetric, as are those we are interested in computing, then one can take advantage of the symmetry to more efficiently compute the matrix.

Let S be the matrix that represents the symmetry of the periodic orbit we are considering. If $X = (x, y, z, v_x, v_y, v_z)$, then $SX = (x, -y, z, -v_x, v_y, -v_z)$ defines the time-reversal symmetry for the halo orbits we are interested in. It is a simple check to prove

Lemma 1. *If $X(t)$ is a solution to $\dot{X} = f(X)$, and $-Sf = f \circ S$, then $SX(-t)$ is also a solution.*

We verified earlier that the CRTBP in rotating coordinates has this symmetry. We now show that the variational equations also have this symmetry.

Lemma 2. *Assume that $-Sf = f \circ S$. Let $F(X) = \partial f / \partial X$ and let $\dot{M} = F(X)M$, then $-SF(X) = F(SX)S$.*

Thus, we have

Lemma 3. *If $(X(t), M(t))$ is a solution to*

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{M} &= F(X)M \ , \end{aligned}$$

then so is $(SX(-t), SM(-t))$.

Now suppose that $X(0)$ is part of a periodic orbit with period $2T$ that is symmetric, i.e., $X(t)$ and $SX(-t)$ are the same orbits. Further, assume that $X(0) = SX(0)$, a point of symmetry. Now let $M(t; X(t_0), M(t_0), t_0)$ denote the solution to the variational equations for the periodic orbit $X(t)$. The monodromy matrix is given by $M(2T; X(0), \mathbf{1}, 0)$. We wish to use the symmetry of the periodic orbit in order to write the monodromy matrix in terms of the solution for only halfway around the orbit, $M(T; X(0), \mathbf{1}, 0)$. We will prove the following

Theorem 4. *[6]*

$$M(2T; X(0), \mathbf{1}, 0) = S \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 2K \end{pmatrix} M(T; X(0), \mathbf{1}, 0)^* \begin{pmatrix} 2K & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} SM(T; X(0), \mathbf{1}, 0) \ ,$$

where the * represents matrix transpose.

Proof. We start by noting that

$$M(2T; X(0), \mathbf{1}, 0) = M(2T; X(t), \mathbf{1}, T) M(T; X(0)\mathbf{1}, 0) \ .$$

Next we use the symmetry to write

$$M(2T; X(T), \mathbf{1}, T) = M(0; X(-T), \mathbf{1}, -T) \ .$$

Since $(X(t; X(T), T), M(t; X(T), \mathbf{1}, T)$ is a solution, then we can apply our symmetry to get a solution $(SX(-t; X(-T), -T), SM(-t; X(-T), \mathbf{1}, -T))$. At time T, this new solution takes on the values $(SX(-T), S)$. Thus by uniqueness of solutions we have the equalities:

$$\begin{aligned} SX(-t; X(-T), -T) &= X(t; SX(-T), T) \\ SM(-t; X(-T), \mathbf{1}, -T) &= M(t; SX(-T), S, T) \ . \end{aligned}$$

Evaluating at $t = 0$ and using the periodicity and symmetry of $X(t)$ we get

$$SM(0; X(-T), \mathbf{1}, -T) = M(0; X(T), \mathbf{1}, T)S \ .$$

So now we have

$$M(2T; X(0), \mathbf{1}, 0) = SM(0; X(T), \mathbf{1}, T)SM(T; X(0)\mathbf{1}, 0)\,.$$

We are in principle done with expressing the monodromy matrix in terms of the fundamental solution matrix evaluated at time T since $M(0; X(T), \mathbf{1}, T) = M(T; X(0)\mathbf{1}, 0)^{-1}$. But we can do a little better than this, since $M(T; X(0)\mathbf{1}, 0)^{-1}$ may be written in terms of $M(T; X(0)\mathbf{1}, 0)^{*}$. This is done by recalling that the fundamental solution matrix for a Hamiltonian system is symplectic [10]. Thus we can perform a change of coordinates and use the symplectic property to write the inverse in terms of a transpose.

Writing M in symplectic coordinates we get

$$\tilde{M} = \begin{pmatrix} \mathbf{1} & 0 \\ -K & \mathbf{1} \end{pmatrix} M \begin{pmatrix} \mathbf{1} & 0 \\ K & \mathbf{1} \end{pmatrix}.$$

For symplectic $\tilde{M}$, we have $\tilde{M}^{-1} = -J\tilde{M}^{*}J$, where $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$. Thus we compute that

$$M^{-1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 2K \end{pmatrix} M^{*} \begin{pmatrix} 2K & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

Putting this all together we get the conclusion of the theorem. □

13 Stable and unstable manifolds of the halo orbits

In this section we give a rough method to approximate the stable and unstable manifolds of the halo orbits. Very briefly, the stable manifold of a periodic orbit is the set of points which converge to the periodic orbit in forward time. Conversely, the unstable manifold consists of those points which converge to the orbit in backwards time. In general, these are manifolds and their smoothness depends on the smoothness of the dynamical system. The idea of our algorithm is to find a linear approximation in the neighborhood of the periodic orbit and then to globalize this approximation by flowing points on it under the vector field.

The stable manifold of a periodic orbit is tangent to the space of eigenvectors of the monodromy matrix whose eigenvalues have magnitude less than one. It is also tangent to the periodic orbit. For the nearly planar halo orbits in the Sun-Earth system, there is only one eigenvalue with magnitude less than one so the stable manifold is two-dimensional. The unstable manifold is also two-dimensional. Eigenvectors of the monodromy matrix are used to construct the local linear approximations. At each point along the trajectory, the eigenvectors of the monodromy matrix may be calculated. A small step in the direction of the appropriate eigenvector will provide an approximation of a point on the stable or unstable manifold.

We note that this method cannot in general be expected to work well. In particular, computational difficulties in computing a good approximation of the stable manifold include

- disparate growth scales,

- poor initial guesses from the linear approximation, and
- an exponential growth of errors.

In general there are difficulties with trying to evolve even a smooth curve in a smooth vector field in that the resulting surface need not be smooth. Some of these issues are discussed in Guckenheimer and Worfolk [8]. For the Sun-Earth system, the stretching away from the periodic orbit is strong enough that the manifold appears to grow nicely. There are complications in the computation of the manifold when it comes close to Earth due to the singularity in the vector field. In the following section we will present some manifolds that were computed using this algorithm.

14 Results

In this section we summarize some of the discoveries our students made in the implementation of the schemes discussed in the earlier sections. For more details on these results, we urge you to examine the online documents of our students [9]. All these calculations were done for the Sun-Earth system which we take to be given by $\mu = 3.03591 \times 10^{-6}$.

The analytic approximation for halo orbits should work well in a neighborhood of the libration points. Since the (out of plane) halo orbits have a minimum amplitude, it is not clear whether the approximation is valid. However for the Sun-Earth system and for halo orbits where $n = 1, 3$, we find that the approximation works very well and provides a convergent seed for our Newton's method for orbits with an out of plane amplitude of up to approximately 0.008 AU ≈ 1.2 million kilometers. In Figure 1 we give one family of halo orbits. The analytic orbits and the numerically converged ones are, on the scale of this figure, indistinguishable. The family of orbits can be numerically continued beyond the size where good analytic approximations are provided. Doing this we find that the orbits become stable, but we have not yet explored further details of the extension of this family.

In the analytic approximation to the CRTBP, there are also halo orbits with a second symmetry given by taking $n = 0, 2$ in Equations 6. Recall that $n\pi/2$ is the phase difference between the oscillation in the x direction and the z direction. For the Sun-Earth system, these orbits are much larger than the others and we have not been able to numerically find any periodic orbits which are similar to these. Thus, they are either an artifact of the truncated analytic model or not well approximated by the analytic model. Further exploration may be able to resolve this question.

Stable and unstable manifolds for class $n = 1, 3$ halo orbits were successfully computed by our students using the method described in Section 13. Visualization of the projection of the manifolds into 3-dimensional position space was provided using Geomview [12]. A few coloring schemes were implemented to indicate stability of the manifolds, and to provide velocity information at each point on the manifold. See the online HTML document [9] for details and pictures. Some examples are given in Figures 2, 3 and 4.

In Figure 2 the halo orbit bounds the open end of the half of the manifold trailing away from the Earth, which is located at the origin of the coordinate axes. L_1 appears as the small sphere at the center of the orbit. A coloring scheme has been chosen so that each dark band on the manifold represents a trajectory asymptotically approaching the halo orbit. Figures 3 and 4 depict

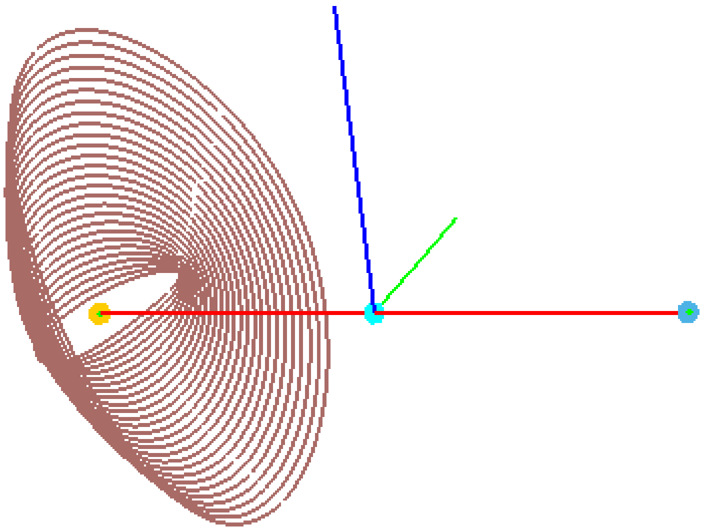

Figure 1: The $n = 1$ family of halo orbits around L_1 in the Sun-Earth system. L_1, the Earth, and L_2 are represented by small spheres.

the Earth-side branch of the stable manifold of the same halo orbit. The manifold passes close to the Earth before doubling back and intersecting itself in physical space (but not phase space!). A single trajectory on the manifold has been superimposed in Figure 4.

The potential for further investigations is great. Future projects of interest include

- a study of the geometry of the surfaces of halo orbits;
- a study of how the stable and unstable manifolds change along the family of halo orbits;
- an exploration of the structure of the manifolds in the proximity of the Earth in a search for low energy transfers from Earth parking orbits to the stable manifold of a halo orbit;
- similar explorations for transfers between halo orbits about L_1 and L_2;
- an exploration of the geometry of the phase space for other mass ratio parameters, say for the Sun-Jupiter system; and
- the computation and visualization of invariant tori in the vicinity of the libration points which give rise to "Lissajous" orbits.

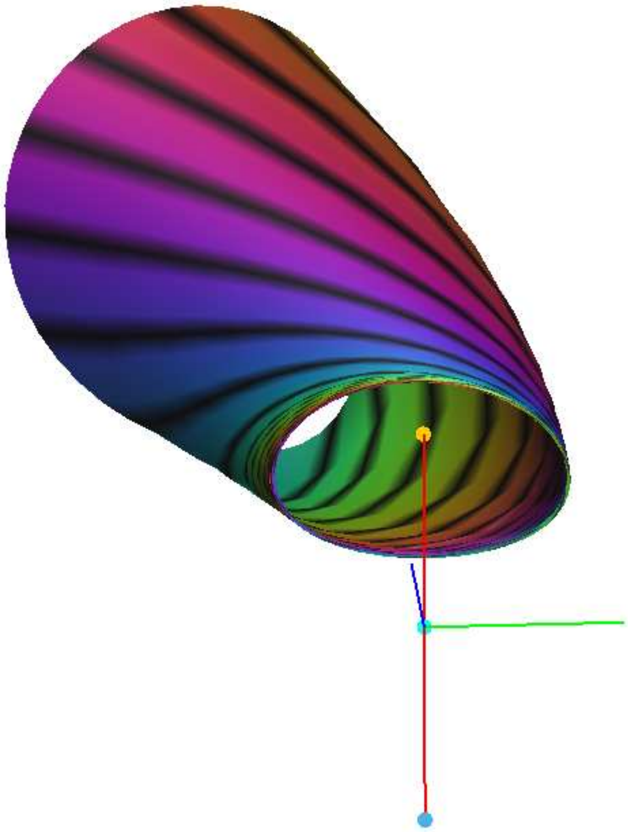

Figure 2: One branch of an L_1 halo orbit's stable manifold, extending away from Earth.

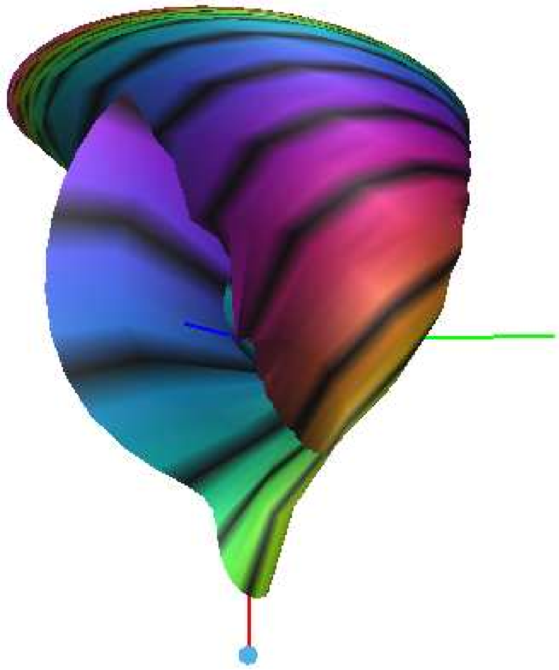

Figure 3: The other branch of the halo orbit's stable manifold, extending towards Earth.

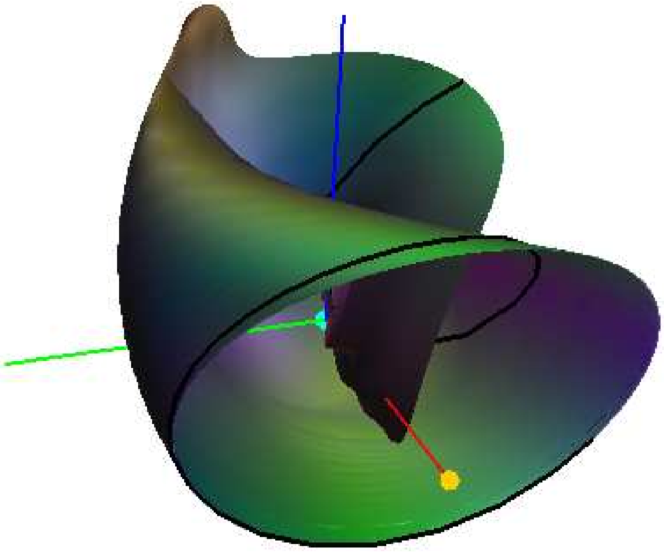

Figure 4: A different view of the same stable manifold, with a trajectory emphasized.

References

[1] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, 1978.

[2] A. Back, J. Guckenheimer, M. Myers, F. Wicklin, and P. Worfolk. dstool: Computer assisted exploration of dynamical systems. *Notices of the Amer. Math. Soc.*, 39(4):303–309, 1992. Software available via anonymous ftp at cam.cornell.edu.

[3] B. T. Barden, K. C. Howell, and M. W. Lo. Application of dynamical systems theory to trajectory design for a libration point mission. preprint, 1996.

[4] V. D. Barger and M. G. Olsson. *Classical Mechanics: A Modern Perspective.* McGraw-Hill, New York, second edition, 1995.

[5] R. R. Bate, D. D. Mueller, and J. E. White. *Fundamentals of astrodynamics.* Dover, New York, 1971.

[6] J. V. Breakwell and J. V. Brown. The 'halo' family of 3-dimensional periodic orbits in the Earth-Moon restricted 3-body problem. *Celestial Mechanics*, 20:389–404, 1979.

[7] H. Goldstein. *Classical Mechanics.* Addison-Wesley, Reading, 2nd edition, 1980.

[8] J. Guckenheimer and P. Worfolk. Dynamical systems: Some computational problems. In D. Schlomiuk, editor, *Bifurcations and Periodic Orbits of Vector Fields*, pages 241–277. NATO ASI Series. Series C: Mathematical and Physical Sciences, Kluwer, 1993.

[9] M. Megraw and C. Sinclair. Orbit visualization project home page. Online HTML document, http://www.geom.umn.edu/∼megraw/home.html, 1996.

[10] K. R. Meyer and G. R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-body Problem.* Springer-Verlag, New York, 1992.

[11] A. H. Nayfeh. *Perturbation Methods.* Pure and applied math. John Wiley & Sons, 1973.

[12] M. Phillips. Geomview manual. Technical Report GCG62, The Geometry Center, University of Minnesota, 1993.

[13] The Royal Astronomical Society of Canada R. L. Bishop, editor. *Observer's Handbook 1996.* University of Toronto Press, 1995.

[14] D. L. Richardson. Periodic orbits about the L_1 and L_2 collinear points in the circular-restricted problem. Technical Report CSC/TR-78/6002, Computer Sciences Corporation, 1978.

[15] D. L. Richardson. Analytic construction of periodic orbits about the collinear points. *Celestial Mechanics*, 22:241–253, 1980.

[16] D. L. Richardson. Halo orbit formulation for the ISEE-3 mission. *J. Guidance and Control*, 3(6):543–548, 1980.

[17] V. Szebehely. *Theory of Orbits.* Academic Press, New York, 1967.

A Astronomical constants

Taken from [13], except where noted.

$$\begin{aligned}
G &= 6.672 \times 10^{-11} m^3 kg^{-1} s^{-2} \\
M_S \text{ (mass of the Sun)} &= 1.9891 \times 10^{30} kg \\
M_E \text{ (mass of the Earth)} &= 5.974 \times 10^{24} kg \\
\mu \text{ (Earth/Sun)} &= 3.0035 \times 10^{-6} \\
\mu \text{ (Earth-Moon/Sun)} &= 3.0404 \times 10^{-6} \\
\mu \text{ (Earth-Moon/Sun)} &= 3.0359 \times 10^{-6} \qquad \text{(from [17])} \\
\mu \text{ (Jupiter/Sun)} &= 3.1463 \times 10^{-3} \\
1\ A.U. &= 1.4960 \times 10^{11} m
\end{aligned}$$

B Solving the inhomogeneous linear equation

In this section we discuss how to solve the inhomogeneous linear equations which arise in the Poincaré-Linstedt method for this problem. First note that the $x - y$ equations and the z equations are completely decoupled, so we can consider them independently.

The $x - y$ equations

Consider the following linear inhomogeneous differential equations with $c_2 > 1$:

$$\begin{aligned}
x'' - 2y' - (1 + 2c_2)x &= A\cos(qt + p) \ , \\
y'' + 2x' + (c_2 - 1)y &= B\sin(qt + p) \ .
\end{aligned}$$

The homogeneous solution is given by

$$\begin{aligned}
x_h(t) &= -C_1 \cos(\lambda t + \phi) + C_2 e^{\alpha t} + C_3 e^{-\alpha t} \ , \\
y_h(t) &= k_2 C_1 \sin(\lambda t + \phi) - k_1 C_2 e^{\alpha t} + k_1 C_3 e^{-\alpha t} \ ,
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \sqrt{\frac{c_2 - 2 + \sqrt{9c_2^2 - 8c_2}}{2}} \ , \\
\lambda &= \sqrt{-\frac{c_2 - 2 - \sqrt{9c_2^2 - 8c_2}}{2}} \ , \\
k_1 &= (2c_2 + 1 - \alpha^2)/2\alpha \ , \\
k_2 &= (2c_2 + 1 + \lambda^2)/2\lambda \ .
\end{aligned}$$

There are two cases for particular solutions, depending on whether on the forcing is at the natural frequency λ or not. Thus, we have

$$\left.\begin{array}{rcl} x_p(t) & = & \frac{2qB-(q^2+1-c_2)A}{q^4+q^2(c_2-2)-2c_2^2+c_2+1}\cos(qt+p) \\ y_p(t) & = & \frac{2qA-(q^2+1+2c_2)B}{q^4+q^2(c_2-2)-2c_2^2+c_2+1}\sin(qt+p) \end{array}\right. \text{, when } q \neq \lambda \text{ ,}$$

and

$$\left.\begin{array}{rcl} x_p(t) & = & \frac{A-kB}{2(\lambda k^2+\lambda-2k)} t\sin(qt+p) \\ y_p(t) & = & k\frac{A-kB}{2(\lambda k^2+\lambda-2k)} t\cos(qt+p) + \frac{k(A(1-\lambda k)-B(\lambda-k))}{2\lambda(\lambda k^2+\lambda-2k)}\sin(qt+p) \end{array}\right. \text{, when } q = \lambda \text{ .}$$

The z equation

For the z equation, we simply have

$$z'' + \lambda^2 z = A\sin(qt+p)$$

The homogeneous solution is given by

$$z_h(t) = C\sin(\lambda t + \psi) \ .$$

Once again there are two cases for the particular solution given by

$$z_p(t) = \frac{A}{\lambda^2 - q^2}\sin(qt+p) \ , \quad q \neq \lambda \ ,$$

and

$$z_p(t) = -\frac{A}{2\lambda} t\cos(qt+p) \ , \quad q = \lambda \ .$$

C Coefficients and quantities from Section 10

$$
\begin{aligned}
\lambda &= \sqrt{\left(2 - c_2 + \sqrt{9c_2^2 - 8c_2}\right)/2} \\
k &= \frac{2c_2 + 1 + \lambda^2}{2\lambda} = \frac{2\lambda}{\lambda^2 + 1 - c_2} \\
\tau_1 &= \lambda\tau + \phi \\
\tau_2 &= \lambda\tau + \psi \\
\alpha_1 &= -\frac{3}{4}c_3[A_x^2(k^2 - 2) + A_z^2] \\
\beta_1 &= \frac{3}{2}c_3 k A_x^2 \\
\beta_3 &= \frac{3}{2}c_3 A_x(\sigma_{21} - k\rho_{21}) - \frac{3}{8}kc_4 A_x^3(4 + k^2) \\
\beta_4 &= \frac{3}{2}c_3 A_x(\sigma_{22} - k\rho_{22}) - \frac{3}{8}c_4 k A_x A_z^2 \\
\beta_5 &= \frac{3}{2}c_3 A_x(\sigma_{22} + k\rho_{22}) + \frac{3}{8}c_4 k A_x A_z^2 \\
\beta_6 &= \nu_2 + 2\omega_2\lambda A_x(\lambda k - 1) + \zeta\beta_5 \\
\gamma_1 &= \frac{3}{4}c_3 A_x^2(2 + k^2) \\
\gamma_2 &= \frac{3}{4}c_3 A_z^2 \\
\gamma_3 &= -\frac{3}{2}c_3 A_x(2\rho_{21} - k\sigma_{21}) - \frac{1}{2}c_4 A_x^3(2 + k^2) \\
\gamma_4 &= \frac{3}{2}c_3(kA_x\sigma_{22} + A_z\kappa_{21} - 2A_x\rho_{22}) - \frac{3}{2}c_4 A_x A_z^2 \\
\gamma_5 &= -\frac{3}{2}c_3(2A_x\rho_{22} + kA_x\sigma_{22} - A_z\kappa_{22}) - \frac{3}{2}c_4 A_x A_z^2 \\
\delta_1 &= \frac{3}{2}c_3 A_x A_z \\
\delta_3 &= -\frac{3}{2}c_3\rho_{22}A_z - \frac{3}{8}c_4 A_z^3 \\
\delta_4 &= \frac{3}{2}c_3(A_x\kappa_{21} - A_z\rho_{21}) - \frac{3}{8}c_4 A_z A_x^2(4 + k^2) \\
\delta_5 &= -\frac{3}{2}c_3(A_x\kappa_{22} - A_z\rho_{21}) + \frac{3}{8}c_4 A_x^2 A_z(4 + k^2)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{21} &= -\delta_1/3\lambda^2 \\
\kappa_{22} &= \delta_1/\lambda^2 \\
\kappa_{31} &= -\frac{\delta_3+\delta_4}{8\lambda^2} \\
\kappa_{32} &= \frac{\delta_3-\delta_4}{8\lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
\nu_1 &= -3c_3(A_x(2\rho_{20}+\rho_{21}+\frac{k}{2}\sigma_{21})+\frac{1}{2}A_z(\kappa_{21}+\kappa_{22})) \\
&\quad +\frac{3}{2}c_4A_x(A_x^2(k^2-2)+2A_z^2) \\
\nu_2 &= \frac{3}{2}c_3A_x(\sigma_{21}-2k\rho_{20}+k\rho_{21})+\frac{3}{2}c_4kA_x[A_x^2(\frac{3}{4}k^2-1)+\frac{1}{2}A_z^2] \\
\nu_3 &= \frac{3}{2}c_3(A_x(\kappa_{21}+\kappa_{22})+A_z(\rho_{22}-2\rho_{20})) \\
&\quad +\frac{3}{2}c_4A_z(\frac{3}{4}A_z^2+(\frac{1}{2}k^2-2)A_x^2)
\end{aligned}
$$

$$
\begin{aligned}
\rho_{20} &= -\alpha_1/(1+2c_2) \\
\rho_{21} &= \frac{4\lambda\beta_1-\gamma_1(4\lambda^2+1-c_2)}{16\lambda^4+4\lambda^2(c_2-2)-2c_2^2+c_2+1} \\
\rho_{22} &= \frac{-\gamma_2(4\lambda^2+1-c_2)}{16\lambda^4+4\lambda^2(c_2-2)-2c_2^2+c_2+1} \\
\rho_{31} &= \frac{6\lambda(\beta_3+\zeta\beta_4)-(9\lambda^2+1-c_2)(\gamma_3+\zeta\gamma_4)}{81\lambda^4+9\lambda^2(c_2-2)-2c_2^2+c_2+1}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{21} &= \frac{4\lambda\gamma_1-\beta_1(4\lambda^2+1+2c_2)}{16\lambda^4+4\lambda^2(c_2-2)-2c_2^2+c_2+1} \\
\sigma_{22} &= \frac{4\lambda\gamma_2}{16\lambda^4+4\lambda^2(c_2-2)-2c_2^2+c_2+1} \\
\sigma_{31} &= \frac{6\lambda(\gamma_3+\zeta\gamma_4)-(9\lambda^2+1+2c_2)(\beta_3+\zeta\beta_4)}{81\lambda^4+9\lambda^2(c_2-2)-2c_2^2+c_2+1} \\
\sigma_{32} &= -\frac{k}{2\lambda}\beta_6 \\
a_1 &= -\frac{3}{2}c_3(2a_{21}-\zeta a_{23}+d_{21}(2-3\zeta))-\frac{3}{8}c_4(8-4\zeta-k^2(2+\zeta)) \\
a_2 &= \frac{3}{2}c_3(a_{24}-2a_{22})+\frac{9}{8}c_4
\end{aligned}
$$

$$
\begin{aligned}
a_{21} &= \frac{3c_3(k^2-2)}{4(1+2c_2)} \\
a_{22} &= \frac{3c_3}{4(1+2c_2)} \\
a_{23} &= -\frac{3\lambda c_3}{4kD_1}(3k^3\lambda - 6k(k-\lambda)+4) \\
a_{24} &= -\frac{3\lambda c_3}{4kD_1}(2+3\lambda k) \\
a_{31} &= -\frac{9\lambda}{D_2}(c_3(ka_{23}-b_{21})+kc_4(1+\frac{1}{4}k^2)) + \frac{9\lambda^2+1-c_2}{2D_2}(3c_3(2a_{23}-kb_{21})+c_4(2+3k^2)) \\
a_{32} &= -\frac{9\lambda}{4D_2}(4c_3(ka_{24}-b_{22})+kc_4) \\
&\quad -\frac{3(9\lambda^2+1-c_2)}{2D_2}(c_3(kb_{22}+d_{21}-2a_{24})-c_4)
\end{aligned}
$$

$$
\begin{aligned}
b_{21} &= -\frac{3c_3\lambda}{2D_1}(3\lambda k-4) \\
b_{22} &= \frac{3\lambda c_3}{D_1} \\
b_{31} &= \frac{1}{D_2}[3\lambda(3c_3(kb_{21}-2a_{23})-c_4(2+3k^2)) \\
&\qquad +(9\lambda^2+1+2c_2)(12c_3(ka_{23}-b_{21})+3kc_4(4+k^2))/8] \\
b_{32} &= \frac{1}{D_2}[3\lambda(3c_3(kb_{22}+d_{21}-2a_{24})-3c_4)+(9\lambda^2+1+2c_2)(12c_3(ka_{24}-b_{22})+3c_4k)/8] \\
b_{33} &= -\frac{k}{16\lambda}(12c_3(b_{21}-2ka_{21}+ka_{23})+3c_4k(3k^2-4)+16s_1\lambda(\lambda k-1)) \\
b_{34} &= -\frac{k}{8\lambda}(-12c_3ka_{22}+3c_4k+8s_2\lambda(\lambda k-1)) \\
b_{35} &= -\frac{k}{16\lambda}(12c_3(b_{22}+ka_{24})+3c_4k)
\end{aligned}
$$

$$
\begin{aligned}
d_{21} &= -\frac{c_3}{2\lambda^2} \\
d_{31} &= \frac{3}{64\lambda^2}(4c_3a_{24}+c_4) \\
d_{32} &= \frac{3}{64\lambda^2}(4c_3(a_{23}-d_{21})+c_4(4+k^2))
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= 16\lambda^4+4\lambda^2(c_2-2)-2c_2^2+c_2+1 \\
D_2 &= 81\lambda^4+9\lambda^2(c_2-2)-2c_2^2+c_2+1 \\
D_3 &= 2\lambda(\lambda(1+k^2)-2k)
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= 2s_1\lambda^2 + a_1 \\
l_2 &= 2s_2\lambda^2 + a_2
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= \frac{1}{D_3}\left\{\frac{3}{2}c_3[2a_{21}(k^2-2) - a_{23}(k^2+2) - 2kb_{21}] - \frac{3}{8}c_4[3k^4 - 8k^2 + 8]\right\} \\
s_2 &= \frac{1}{D_3}\left\{\frac{3}{2}c_3[2a_{22}(k^2-2) - \zeta a_{24}(k^2+2) - \zeta 2kb_{22} + d_{21}(2-3\zeta)]\right. \\
&\quad \left. + \frac{3}{8}c_4[(8-4\zeta) - k^2(2+\zeta)]\right\}
\end{aligned}
$$

D Solutions to exercises

Solution 2.1

If

$$U(x) = \frac{-k}{|X|} = \frac{-k}{(\sum X_i^2)^{1/2}} ,$$

then

$$\partial X/\partial X_i = -\frac{1}{2}\frac{-k}{(\sum X_i^2)^{3/2}} 2X_i = \frac{kX_i}{|X|^3} .$$

Thus

$$-\nabla U(X) = -\frac{kX}{|X|^3} .$$

Solution 2.2

Let $u = X$ and $v = \dot{X}$, then

$$
\begin{aligned}
\dot{u} &= v \\
\dot{v} &= -ku/|u|^3 .
\end{aligned}
$$

Written component-wise, this is a system of 6 first-order ordinary differential equations.

Solution 2.3

$$
\begin{aligned}
\frac{d}{dt}(X(t) \times \dot{X}(t)) &= \dot{X}(t) \times \dot{X}(t) + X(t) \times \ddot{X}(t) \\
&= 0 + 0 .
\end{aligned}
$$

This implies that $X \times \dot{X}$ is a constant of the motion and thus the motion is in a plane which has normal $X \times \dot{X}$.

Solution 2.4

Substituting $X(t) = (a\cos\omega t, a\sin\omega t, 0)$ into the equation of motion we compute:

$$-a\omega^2 \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} = -\frac{ka}{a^3} \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} .$$

This equation is satisfied when

$$\omega^2 = \frac{k}{a^3} .$$

Solution 3.1

Since $r = X_2 - X_1$,

$$\begin{aligned} \ddot{r} &= \ddot{X}_2 - \ddot{X}_1 \\ &= -(Gm_1 + Gm_2)r/|r|^3 . \end{aligned}$$

This is the central force motion equation with $k = G(m_1 + m_2)$.

Solution 3.2

In Exercise 2.4, we found that $\omega = \sqrt{k/a^3}$. Using $k = G(m_1 + m2) \approx 6.672 \times 10^{-11} \cdot 1.9891 \times 10^{30}\ \ m^3 s^{-2}$ and $a = 1.496 \times 10^{11} m$, we find that

$$\omega \approx 1.991 \times 10^{-7} s^{-1}$$

and the period $T = 2\pi/\omega$ is

$$T \approx 3.156 \times 10^7 sec \approx 365.26 \text{ days} .$$

Solution 3.3

The center of mass r_0 is given by solving the following equation:

$$(m_1 + m_2)r_0 = m_1 X_1 + m_2 X_2 .$$

Thus,

$$\ddot{r_0} = (m_1\ddot{X}_1 + m_2\ddot{X}_2)/(m_1 + m2) .$$

Substituting in the equations of motion for X_1 and X_2 we find that $\ddot{r}_0 = 0$. Thus we can integrate and solve explicitly to get $r_0(t) = at + b$, where a and b and the initial velocity and position, respectively.

Solution 3.4

Since $r_0 = 0$, we have $m_1 X_1 + m_2 X_2 = 0$. By definition, $r = X_2 - X_1$. We first eliminate X_2 to find

$$X_1 = -\frac{m_2}{M} r ,$$

and then find

$$X_2 = \frac{m_2}{M} r \ .$$

Solution 4.1

The potentials U_1 and U_2 from the masses m_1 and m_2 located at X_1 and X_2, respectively, acting on a mass m located at X are given by

$$U_1 = -Gm_1 m/|X - X_1| \ , U_2 = -Gm_2 m/|X - X_2| \ .$$

The full potential U is given by $U = U_1 + U_2$. The force on the mass m is given by $F = -\nabla U = -\nabla U_1 - \nabla U_2$.

Solution 4.2

Let $X_1(t) = a(\cos \omega t, \sin \omega t, 0)$ and $X_2(t) = -b(\cos \omega t, \sin \omega t, 0)$. Then

$$\ddot{X} = -\frac{Gm_1}{|X - X_1(t)|^3}(X - X_1(t)) - \frac{Gm_2}{|X - X_2(t)|^3}(X - X_2(t)) \ .$$

Solution 4.3

Recall that $l = a + b$ and $M = m_1 + m_2$. Because the center of mass is at the origin, we have the relation $am_1 = bm_2$. Then

$$\frac{b}{l} = \frac{m_1 a/m_2}{a + b} = \frac{m_1 a}{m_2 a + m_1 a} = \frac{m_1}{M}$$

and

$$\frac{a}{l} = \frac{m_2 b/m_1}{a + b} = \frac{m_2 b}{m_2 b + m_1 b} = \frac{m_2}{M} \ .$$

Solution 4.4

Recall that $R_t^{-1} = R_t^T = R_{-t}$. Let $W = R_t X$, then

$$\begin{aligned} \dot{W} &= \dot{R}_t X + R_t \dot{X} \\ &= \dot{R}_t R_{-t} W + R_t \dot{X} \end{aligned}$$

and solving for $\dot{X}$ we find

$$\dot{X} = R_{-t}\dot{W} - R_{-t}\dot{R}_t R_{-t} W \ .$$

Next,

$$\begin{aligned} \ddot{W} &= \ddot{R}_t X + \dot{R}_t \dot{X} + \dot{R}_t \dot{X} + R_t \ddot{X} \\ &= \ddot{R}_t R_{-t} W + 2(\dot{R}_t R_{-t} \dot{W} - \dot{R}_t R_{-t} \dot{R}_t R_{-t} W) + R_t(-\nabla_X \phi(X, t)) \\ &= (\ddot{R}_t R_{-t} - 2\dot{R}_t R_{-t} \dot{R}_t R_{-t}) W + 2\dot{R}_t R_{-t} \dot{W} \\ &\quad -R_t (R_{-t}^T)^{-1} \nabla_W \phi(R_{-t} W) \ . \end{aligned}$$

Now let

$$K = \dot{R}_t R_{-t} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

and notice that

$$-(\ddot{R}_t R_{-t} - 2\dot{R}_t R_{-t} \dot{R}_t R_{-t}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = K^2 .$$

Thus, letting $U(W) = \phi(R_{-t}W, t)$ we find that

$$\ddot{W} - 2K\dot{W} + K^2 W = -\nabla U(W) .$$

Solution 5.1

The libration points are stationary points and are solutions to $\nabla\Omega = 0$. This is equivalent to

$$\begin{aligned} x - U_x &= 0 \\ y - U_y &= 0 \\ U_z &= 0 . \end{aligned}$$

Let $\rho_1 = |X - (1-\mu)e_1|$ and $\rho_2 = |X + \mu e_1|$ be the distances from the primary masses to X. Then the above equations for the equilibrium points are

$$\begin{aligned} x - \frac{\mu(x-(1-\mu))}{\rho_1^3} - \frac{(1-\mu)(x+\mu)}{\rho_2^3} &= 0 \\ y - \frac{\mu y}{\rho_1^3} - \frac{(1-\mu)y}{\rho_2^3} &= 0 \\ -\frac{\mu z}{\rho_1^3} - \frac{(1-\mu)z}{\rho_2^3} &= 0 . \end{aligned}$$

From the third equation, we discover that $z = 0$. From the second, we find there are two types of solutions, one type for $y \neq 0$ and another for $y = 0$. We will first solve for the solutions with $y \neq 0$.

For $y \neq 0$,

$$1 - \mu/\rho_1^3 - (1-\mu)/\rho_2^3 = 0 .$$

We use this equation to eliminate ρ_2 from the first equation, thus discovering that $\rho_1 = 1$. Using this, we find that also $\rho_2 = 1$. Thus the two points which are unit distance from both masses are libration points. Since the masses are unit distance apart, these libration points lie on the vertices of an equilateral triangle whose other two vertices are the locations of the masses. Solving explicitly, we find they are given by $(1/2 - \mu, \pm\sqrt{3}/2, 0)$.

When $y = 0$, then we are looking for stationary points which lie on the line connecting the two masses — the collinear libration points. We are reduced to solving for x in the following equation:

$$x - \frac{\mu(x-(1-\mu))}{|x-(1-\mu)|^3} - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} = 0 .$$

By considering the three different regions separated by the location of the primary masses

$$\begin{array}{c} -\mu < x < 1-\mu \\ 1-\mu < x \\ x < -\mu \ , \end{array}$$

this equations may be written as

$$x - s_1 \frac{\mu}{(x-(1-\mu))^2} - s_2 \frac{1-\mu}{(x+\mu)^2} = 0 \ ,$$

where the s_i have the following values in the three regions, respectively,

$$\begin{array}{lllllll} s_1 & = & -1, & s_2 & = & 1 \\ s_1 & = & 1, & s_2 & = & 1 \\ s_1 & = & -1, & s_2 & = & -1 \ . \end{array}$$

This in turn leads to solving a fifth-order algebraic equation (for each region).

Solution 6.1

Let

$$\mathcal{J} = 2\Omega(x,y,z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ .$$

We compute

$$\begin{array}{rcl} d\mathcal{J}/dt & = & 2 < \nabla_X \Omega, \dot{X} > -2 < \ddot{X}, \dot{X} > \\ & = & 2 < \nabla_X \Omega - \ddot{X}, \dot{X} > \\ & = & 2 < -(2\dot{y}, -2\dot{x}, 0), (\dot{x}, \dot{y}, \dot{z}) > \\ & = & 0 \ . \end{array}$$

So $\mathcal{J}$ is an integral.

Solution 6.2

The following Maple code does the job.

```
with(plots):
mu := 0.25;
rho1 := sqrt((x-(1-mu))\^2+y\^2+z\^2);
rho2 := sqrt((x+mu)\^2+y\^2+z\^2);
U := -(mu/rho1+(1-mu)/rho2);
Omega := 0.5*(x\^2+y\^2)-U;
C := 2.0;
implicitplot3d(Omega=C/2, x=-2..2, y=-2..2, z=-2..2,
               title=cat(`Mu=`,convert(mu,string),`, C=`,convert(C,string)),
               axes=boxed);
```

Solution 7.1

Assuming that $(x(t), y(t), z(t))$ satisfies the equations of motion, we must show that the symmetric solutions do also. Doing this for $(x(t), y(t), -z(t))$ we get

$$\begin{aligned} \ddot{x}(t) - 2\dot{y}(t) &= \Omega_x(x(t), y(t), -z(t)) \\ \ddot{y}(t) + 2\dot{x}(t) &= \Omega_y(x(t), y(t), -z(t)) \\ -\ddot{z}(t) &= \Omega_z(x(t), y(t), -z(t)) \ . \end{aligned}$$

Since

$$\begin{aligned} \Omega_x(x(t), y(t), -z(t)) &= \Omega_x(x(t), y(t), z(t)) \\ \Omega_y(x(t), y(t), -z(t)) &= \Omega_y(x(t), y(t), z(t)) \\ \Omega_z(x(t), y(t), -z(t)) &= -\Omega_z(x(t), y(t), z(t)) \ , \end{aligned}$$

the above equations hold, and $(x(t), y(t), -z(t))$ is indeed a solution.

Similarly substituting $(x(-t), -y(-t), z(-t))$ into the equations of motion, we get

$$\begin{aligned} \ddot{x}(-t) - 2\dot{y}(-t) &= \Omega_x(x(-t), -y(-t), z(-t)) \\ -\ddot{y}(-t) - 2\dot{x}(-t) &= \Omega_x(x(-t), -y(-t), z(-t)) \\ \ddot{z}(-t) &= \Omega_z(x(-t), -y(-t), z(-t)) \ . \end{aligned}$$

Now,

$$\begin{aligned} \Omega_x(x(-t), -y(-t), z(-t)) &= \Omega_x(x(-t), y(-t), z(-t)) \\ \Omega_x(x(-t), -y(-t), z(-t)) &= -\Omega_x(x(-t), -y(-t), z(-t)) \\ \Omega_z(x(-t), -y(-t), z(-t)) &= \Omega_z(x(-t), y(-t), z(-t)) \ , \end{aligned}$$

so the equations hold (at time -t), and $(x(-t), -y(-t), z(-t))$ is a solution.

We verify the third symmetry in the same fashion.

Solution 8.1

For the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - U(x),$$

we have, from the Euler-Lagrange equations,

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \\ &= \frac{d}{dt}\dot{x} - \nabla U(x) \\ &= \ddot{x} - \nabla U(x), \end{aligned}$$

which is the same as the Newton formulation.

Solution 8.2

The Lagrangian in rotating coordinates is

$$L = \frac{1}{2}(\dot{W} - KW)^2 - U(W).$$

The equations of motion are given by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{W}} - \frac{\partial L}{\partial W} = 0.$$

We have

$$\begin{aligned} \frac{\partial L}{\partial \dot{W}} &= \dot{W} - KW \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{W}} &= \ddot{W} - K\dot{W} \\ \frac{\partial L}{\partial W} &= (\dot{W} - KW)(-K) - \nabla U(W). \end{aligned}$$

Now use the fact that $K^T = -K$ to see

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{W}} - \frac{\partial L}{\partial W} \\ &= \ddot{W} - K\dot{W} + \dot{W}K - KWK + \nabla U \\ &= \ddot{W} - 2K\dot{W} + K^2 W + \nabla U, \end{aligned}$$

which are the equations of motion we know and love.

Solution 9.1

Let $p = \partial L/\partial \dot{q}$, where $L(q, \dot{q})$ is the Lagrangian. Then set $H(p, q) = < p, \dot{q} > - L$ and compute

$$\partial H/\partial p = \dot{q}.$$

So the first Hamiltonian equation of motion $\dot{q} = \partial H/\partial p$ is trivially satisfied.

We then compute

$$\partial H/\partial q = -\partial L/\partial q$$

and see that the second Hamiltonian equation of motion $\dot{p} = -\partial H/\partial q$ becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q},$$

which is the Lagrangian form of the equations of motion.

Solution 9.2

The Jacobian integral is

$$\mathcal{J}(q, \dot{q}) = 2\Omega(q) - \dot{q}^2 = - < K^2 q, q > -2U(q) - \dot{q}^2.$$

The Hamiltonian is

$$\begin{aligned}H(q,\dot q) &= \frac{1}{2}(\dot q - Kq)^2 + <\dot q - Kq, Kq> + U(q)\\ &= \frac{1}{2}\dot q^2 - <\dot q, Kq> + \frac{1}{2} <Kq,Kq> + <\dot q, Kq> - <Kq,Kq> + U(q)\\ &= \frac{1}{2}\dot q^2 - \frac{1}{2} <Kq,Kq> + U(q)\\ &= \frac{1}{2}\dot q^2 + \frac{1}{2} <K^2 q, q> + U(q)\ ,\end{aligned}$$

and we see $H = -\mathcal{J}/2$.

Solution 10.1 (Solution due to Molly Megraw.)

Let $x = \alpha_i$ be the x-coordinate of the collinear libration point L_i. Then x solves

$$x - \frac{\mu}{|x-\mu'|^3}(x-\mu') - \frac{\mu'}{|x+\mu|^3}(x+\mu) = 0,$$

where $\mu' = 1-\mu$. Set

$$S_1 = \tfrac{\mu}{|x-\mu'|^3}, \quad S_2 = \tfrac{\mu'}{|x+\mu|^3}.$$

Then

$$S_1 = \frac{x - S_2(x+\mu)}{x-\mu'},$$

and we are trying to show $S_1 + S_2 > 1$. But

$$\begin{aligned}S_1 + S_2 &> 1 \quad \Longleftrightarrow\\ S_2\left(1 - \frac{x+\mu}{x-\mu'}\right) &> 1 - \frac{x}{x-\mu'} \quad \Longleftrightarrow\\ S_2 \frac{-\mu' - \mu}{x-\mu'} &> \frac{-\mu'}{x-\mu'} \quad \Longleftrightarrow\\ S_2 \frac{-1}{x-\mu'} &> \frac{-\mu'}{x-\mu'}.\end{aligned}$$

Now if x is at L_2, then $x - \mu' > 0$, and the above holds if and only if

$$\begin{aligned}S_2 &< \mu' \quad \Longleftrightarrow\\ \frac{1}{|x+\mu|^3} &< 1 \quad \Longleftrightarrow\\ |x+\mu| &> 1.\end{aligned}$$

But this holds, since the distance between the primaries is 1, and the mass at $1-\mu$ separates L_2 from the mass at $-\mu$. The inequalities above are reversed for x at L_1, leading to the inequality $|x+\mu| < 1$. This inequality holds trivially, since L_1 is between the primaries.

Corrections list for Richardson's work

In this section we list the differences between the formulae presented here and those in the original work of Richardson [14].

Typographical errors

These first corrections to [14] appear to simply be typos. We shall present them here for completeness.

1. The first term in the formula for δ_3 is missing a factor of 1/2. This also shows up in the parameter d_{31}. The formula for d_{31} is corrected in the journal paper [15].
2. The expressions for a_{32} and b_{32} have a factor of ζ which should not be there. The journal paper only uses $\zeta = -1$ and thus does not have this factor; the expressions for a_{32} and b_{32} are given correctly.
3. In equations 2-65, the first occurrence of ρ_{21} should be ρ_{20}.
4. In equations 2-68, the term $\cos 3\tau_1$ with coefficient $d_{21} A_x A_x$ should be $\cos 2\tau_1$.
5. In equations 2-65 and 2-68, the term $\sin \tau_1$ in the z equation should switch between sin and cos with various signs depending on the value of n.

Removal of the third order secular terms

The next corrections are due to the fact that the secular terms in the third-order equations were removed incorrectly. This affects both the third order equations and the third order solutions. The reader can easily refer back to the text to see the differences between the expression in this paper and the original [14]. Essentially, this results in an additional term with coefficient σ_{32} being added to the expression for y_3, which in turn results in third order corrections to the amplitude of $\sin \tau_1$ in the final expression for $y(\tau)$ (with new coefficients b_{33}, b_{34}, b_{35}).