Lab #4

The purpose of the lab is to understand gradient vector fields and their relationship with higher-dimensional surfaces and level sets of those surfaces. In particular, to explore the geometry and discover the relationships between a surface, gradients to that surface, and the "gradient flow" on that surface.

Background: One of the greatest acheivements of Newton was formulating the mathematics of gravitational attraction. Newton published his laws of gravitation in his famous book, *Principia Mathematica*, in 1687. We will not subject you to Newton's original formulation of gravitational laws as it appeared in latin, but instead summarize in English: two objects separated by a distance r feel a gravitational force due to each other. The magnitude of this force is proportional to $1/r^2$.

It turns out that the gravitational force field is a so-called "gradient field." This means that the force at any point is actually the gradient of some function, called the "gravitational potential function." An important way to visualize gravitational force fields is to visualize the graph of the gravitational potential function.

In order to simplify the mathematics, we assume that we are only concerned about the gravitational field in a plane. This is actually a close approximation to reality, since most of the objects in the solar system (with the exception of pluto and some comets) essentially lie in a plane (called the "ecliptic plane"). Also, since we are concentrating on the geometry, we will scale the physical equations so that we do not need to worry about physical quantities such as the mass of the earth or the distance from the earth to a satellite or to the moon.

We begin by considering the gravitational force felt by a satellite in space. We place the earth at the center of our coordinate system. We then define the gravitational potential function to be

$$P(x, y) = -\frac{1}{\sqrt{x^2 + y^2}}$$

and the force felt by an object located at position (x, y) to be

$$F = -\nabla P.$$

To get started, launch Maple and type in the following three commands:

```
with(linalg):  with(plots):
P := -1/sqrt(x^2 + y^2);
F := grad( -P, [x,y] );
```

Note that we have defined the force to be the "minus gradient" of the potential function. Notice also that the the potential is a function from $R^2 \to R$, whereas the force is vector-valued.

(Q1) Compute the magnitude of the force. Is it the case that the magnitude is proportional to $1/r^2$ where r is the distance from the object to the origin?

Activity: Use Maple to visualize the graph of the potential function on the domain $[-1,1]\times[-1,1]$. Hint: you may want to add the option `view= -4..0` to your `plot3d` command. (See `?plot3d[options]` to read about what this option does.)

Now think about placing a ball on the graph of the potential function. When the ball is far away from the origin, the potential surface is only slightly tilted, so the ball slowly rolls towards the origin. If you are near the origin, however, the potential surface is very steep, so the ball feels a very strong pull towards the origin. Mathematically, we say that the force felt by an object is the minus gradient of the potential function. Let's see how this all works:

(Q2) Answer the following questions:

1. Compute the force (a vector, call it v) felt by a particle at location $(x, y) = (1, 1)$.
2. Think of v as being based at the point $(x, y) = (1, 1)$. What direction does v point and what is its magnitude?
3. Suppose I tell you that an object feels a force of $(8\sqrt{5}/25, -4\sqrt{5}/25)$. Either determine where the object is located *or* explain why there is more than one location that the object could be.

(Q3) Use Maple command

```
gradplot( -P, x=-1..1, y=-1..1, grid=[5,5]);
```

to visualize the "vector field" F at a few locations. How does the plot of these vectors relate to the graph of the potential function that we saw earlier? Mention both the magnitude and the direction of the vectors.

(Q4) In class we related the gradient vector to level sets of the function. Use the `contourplot` command to look at the contours of the potential function, or interactively change the previous 3D plot to show contours. (Hint: again, you may want to tag on the option `view=-4..0` inside your Maple command. How do the contours relate to the gradient vectors that we computed earlier?

In a few weeks we will study differential equations and will revisit this example. For now:

(Q5) Explain (using your knowledge of the physical situation) what would happen if an object was placed somewhere in the gravitational force field we've been looking at. Assume the object starts at rest, and describe how its position, velocity, and acceleration changes as time increases. We don't need any sophisticated formulas here, just describe in words and/or pictures what you think will happen. Conjecture what would happen if the object does not start at rest.

For this last question, we will change the scenario under consideration and ask you to investigate the new situation.

Assume that instead of one gravitational mass at the origin, we have two gravitational masses. The one mass (say, the earth) is located at $(x, y) = (-1, 0)$ and is very massive; the other mass (say, the moon) is located at $(x, y) = (0, 1)$ and is less massive. In actuality, the moon has about 1/100 the mass of the earth; we will use a "massive moon" in order to make the geometry more apparent. In particular, we will model the geometry of this new situation by the gravitational potential function and the new force field (in Maple notation) by

```
P := -1/sqrt((x+1)^2 + y^2) - 1/10/sqrt((x-1)^2 + y^2);
F := grad( -P, [x,y] );
```

(Q6) Your task is to use Maple to investigate the gravitational potential function for this new scenario. (Hint: You may want to choose your plotting range differently than for the previous example.) Explain what the potential surface looks like and what the gravitational force field looks like. Also, conjecture as to how an object (initially at rest) will move in this force field. Describe both how its position and velocity will change (a sketch of "position versus time" and "velocity versus time" might be useful, or use a parametrized curve with velocity vectors). Does the object's ultimate fate depend on where the object begins? Is there any location where the object could be placed where it would not move at all?