Chapter 6

Models

This manual is available on-line at the URL

`http://www.geom.umn.edu/locate/pisces/`

The on-line version may be more current, and may contain features not contained in the printed version. For example an the on-line version contains an interactive interface to Pisces via the World-Wide Web.

6.1 Models

- Definitions for Models distributed with Pisces
  - Pisces Function
  - Singular Pisces Function
  - Cubic 2D
  - Bifolium
  - Bow
  - Tacnode
  - Apparent Contour of Torus
  - Fourleaved Rose

  - Threeleaved Rose
  - Twoleaved Rose
  - Intersection of Cylinder and Sphere
  - 3D Cubic
  - 3D Cox-Little-Oshea
  - 3D Quadratic
  - 3D Torus
  - DS Arnold Tongue
  - DS Quadratic
- How to install a new model into Pisces.

6.2 DS Arnold Tongue

A one-dimensional mapping of the circle to itself defined by

f(x)= x + w + epsilon/(2 Pi) sin(2 Pi x)

The set of points for which *f(x)-x=0* are the fixed points for this mapping. The set of points for which *f(x)-x = f'(x)-1 = 0* are called points of saddle-node bifurcation. The set of parameter values leading to fixed points is called the resonance region for the map. See Chapter 6 of J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.

Since this model has a one-dimensional domain, one or more of the parameters must be permuted into the domain before attempting to trace level sets.

6.3 Bifolium

The bifolium is a classical plane curve given by

*f(x,y) = (x^2 + y^2)^2 -a*x^2*y.*

In polar coordinates, this curve is given parametrically as

r(theta) = a sin(theta) [cos(theta)]^2.

The parameter *a* scales the level set, but does not change its topology.

6.4 Bow

The definition for this model is

*f(x,y)= x^3*y - x*y^3 +x^5 +2*y^5*

The level set is called a "bow" because it looks like a bow tied with string on a package. The singular point on this level set is particularly nasty, having eight tangent directions.

6.5 3D Cox-Little-Oshea

A fourth-degree equation in three variables:

f(x,y,z)= A x^2 + B y^2 z^2 + C z^3

The level set of this function with *A=1, B= -1, C=1* adorns the front cover of *Ideals, Varieties, and Algorithms* by D. Cox, J. Little, and D. O'Shea, Springer, 1992. The level set contains an entire curve of singularities.

6.6 Cubic 2D

A general family of cubic polynomials in x and y:

f(x,y)= A10 x + A01 y + A20 x^2 + A11 xy + A02 y^2 + A30 x^3 + A21 x^2 y + A12 xy^2 + A03 y^3

Each parameter sets the value of the coefficient of a different term. In the Pisces user interface, *A[i,j]* is called `coef_x^{i}y^{j}`.

6.7 3D Cubic

A family of cubic polynomials in three variables:

f(x,y,z)= A1 x + A2 x^2 + A3 x^3 + B1 y + B2 y^2 + C1 z + C2 z^2 + D x y

Each parameter sets the value of the coefficient of a different term. In the Pisces user interface, the coefficients are known by names such as `coef_x^2`.

This is **not** a general cubic polynomial in three variables.

6.8 Intersection of Cylinder and Sphere

This model is a curve formed by the intersection of a cylinder and a sphere. The curve is the solution set to the equations *f(x,y,z) = x^2 + y^2 + z^2 - sphere_rad^2*
g(x,y,z) = (x-cyl_xcenter)^2 + y^2 - cyl_rad^2

The parameters `sphere_rad^2, cyl_xcenter`, and `cyl_rad` control the radius of the sphere, the position of the cylinder's axis, and the radius of the cylinder, respectively.

6.9 Fourleaved Rose

This model is the level set of *f(x,y) = (x^2 + y^2)^3 - 4 a x^2 y^2*

The singularity at the origin for this function has eight tangent directions. The function is extremely flat near the singularity, so this model is a good test-case for singularity-detection.

This function can also be defined in the polar form as *r=a sin(2 theta)*

6.10 Pisces Function

This function has an interesting level set made up by two fish ("pisces").

f(x,y) = fish * eye1 * eye2

The function is composed of these three parts:

fish = (x^3)/3 - x + alpha*y*(-1+ (alpha*y)^2*(1/6 + (a*y)^2*(-1/120 + (alpha*y)^2/5040)))

eye1 = (x - xeye)^2 + (y - .5 - yeye)^2 - eyerad^2

eye2 = (x + xeye)^2 + (y + .5 - yeye)^2 - eyerad^2

where x and y are the domain variables,

- xeye and yeye set the position of the eyes
- eyerad controls the radius of the fish eyes
- alpha controls the scale *(pun intended)* of the fish body

6.11 Singular Pisces Function

This function has an interesting level set made up by two fish ("pisces"), but also contains five singular points.

f(x,y) = fish1 * fish2

The function is composed of these two parts:

fish1 = (x*x*x - (y-alpha)*(y-alpha) - x +beta)* ((x+xeye)*(x+xeye) + (y-alpha-yeye)*(y-alpha-yeye) - radeye*radeye);

fish2= (-x*x*x - (y+alpha)*(y+alpha) + x +beta)* ((-x+xeye)*(-x+xeye) + (y+alpha-yeye)*(y+alpha-yeye) - radeye*radeye);

where x and y are the domain variables,

- xeye and yeye set the position of the eyes
- eyerad controls the radius of the fish eyes
- alpha and beta controls the scale *(pun intended)* of the fish body

6.12 DS Pitchfork

A one-dimensional vector field on the line defined by

f(x)= x^3 - mu x + alpha + beta mu

The points for which *f(x)=0* are the equilibria for this vector field. The set of points for which *f(x) = f'(x) = 0* are called points of saddle-node bifurcation. See Chapter 3 of J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.

Since this model has a one-dimensional domain, one or more of the parameters must be permuted into the domain before attempting to trace level sets.

6.13 3D Quadratic

A general quadratic equation in three variables:

f(x,y,z)= A100 x + A200 x^2 + A010 y + A020 y^2 + A001 z + A002 z^2 + A110 x y + A101 x z + A011 y z

This family includes planes, ellipsoids, hyperboloids, cones, cylinders, etc.

6.14 DS Quadratic

A one-dimensional mapping of the interval to itself defined by *f(x)= x^2 + b x + c*

When *b=0*, this is the famous quadratic mapping of the real interval. (See R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin-Cummings, 1986.

The set of points for which *f(x)-x=0* are the fixed points for this mapping. The set of points for which *f(x)-x = f'(x)-1 = 0* are called points of saddle-node bifurcation.

Since this model has a one-dimensional domain, one or more of the parameters must be permuted into the domain before attempting to trace level sets.

6.15 Tacnode

This model has two singularities: a *tacnode* singularity at the origin, and a regular node singularity. The model is defined by the equation

*f(x,y) = 2*x^4 - 3*x^2*y +y^4 -2*y^3 +y^2*

6.16 Threeleaved Rose

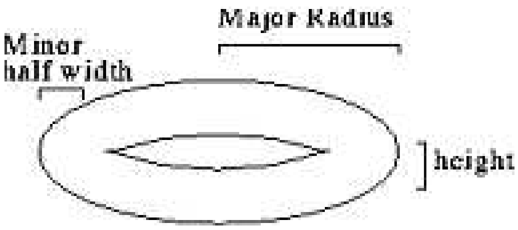
This model is the level set of *f(x,y) = (x^2 + y^2)^2 - 4 a y x^2 + a y (x^2+y^2)*

The singularity at the origin for this function has six tangent directions. The function is extremely flat near the singularity, so this model is a good test-case for singularity-detection.

This function can also be defined in the polar form as *r= a sin(3 theta)*

6.17 3D Torus

The level set for this model is a torus. The user can control the breadth, thickness, and height of the torus. The underlying implicit equation is a fourth degree polynomial in three variables.



6.18 Apparent Contour of Torus

This model is the apparent projection of a torus onto the plane. Consider a translucent torus positioned over a plane with light shining through it. The apparent contour is the shadow of the projection. Notice that some points on the plane have four points of the torus between them and the light source, others have only two points between them, while other points have none. The apparent contour separates the plane into regions with the above property.

The equation of the apparent contour model is very complicated. We start with a geometric torus in 3-space whose "hole" is in the z-direction. We then rotate the torus azimuthally by *theta* radians. We then do a (Grobner basis) computation that gives us the set of *(x,y)* values such that there exists some z-value so that *(x,y,z)* is on the torus and there exists a vertical tangent vector to the torus at *(x,y,z)*.

The model has three parameters: `small_radius` is the height of of the torus (the "thickness" of an bagel); `big_radius` is the major radius of the torus (the distance from the "hole" of a bagel to the middle of the doughy portion); and `theta` is the angle of rotation from the horizontal *(-Pi ¡ theta ¡ Pi)*

6.19 Twoleaved Rose

This model is the level set of $f(x,y) = (x^2 + y^2)^2 - a^2 (x^2-y^2)$

The singularity at the origin for this function has four tangent directions.

This function can also be defined in the polar form as $r^2 = a^2\ \mathit{sin}(2\ \mathit{theta})$