Fixed Points due to Gravitational Lenses

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Abstract. A fixed point of a gravitational lensing map represents those positions from which a pointlike light source has a lensed image that, despite gravitational lensing, corresponds to the original position of the source. This paper studies fixed points of lensing maps due to a generic gravitational lens with applications to nonsingular isolated lenses, and to point-mass lenses with continuous matter and shear. Counting formulas and bounds on the number of fixed points are determined. The results include an Odd-Number Fixed-Point Theorem for nonsingular isolated deflectors. Information on the positions of fixed points are found for the case of point masses on a lens plane with and without shear. The methods of the paper are based on Morse theory, complex variables, and resultants.

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I. INTRODUCTION

Consider a light source at y in R^2 undergoing gravitational lensing by a matter distribution with Newtonian potential ψ. Associated with ψ is a planar lensing map defined by $\boldsymbol{\eta}(\mathrm{x}) = \mathrm{x} - grad\,\psi(\mathrm{x})$. The lensed images of a light source at y are solutions x of the lens equation: $\boldsymbol{\eta}(\mathrm{x}) = \mathrm{y}$. Due to light deflection, a lensed image of a light source does not represent the actual position or magnification of the source. We are interested in studying those positions from which a light source has a lensed image that, in spite of a deflector, coincides with the original position of the light source. They are "fixed points" due to the lens. Formally, a *fixed point* of $\boldsymbol{\eta}$ is a point x_0 such that $\boldsymbol{\eta}(\mathrm{x}_0) = \mathrm{x}_0$. Note that by construction, fixed points of $\boldsymbol{\eta}$ are critical points of ψ. To the best of our knowledge, these points have not been studied before in gravitational lensing. In addition, given the fundamental role of this notion in mathematics, fixed-point lensed images warrant investigation, at least on mathematical grounds. Fixed points are vital to the field of discrete dynamical systems, and the planar mappings arising in gravitational lensing may be analyzed using geometric techniques from this field.

We will show that fixed points not only exist, but obtain counting formulas and lower bounds, as well as some results on their positions. Our methods employ the Morse-theoretic and Resultant techniques in the recent papers of Petters[1,2] and Petters & Witt[3]. Some of these results were announced at the Eighth Marcel Grossman Meeting on General Relativity.

The outline of our paper is as follows. Section II presents counting formulas and lower bounds on the number of fixed points due to a "generic" gravitational lens. The remainder of the paper then deals with applications to nonsingular isolated lenses (Section III), point masses (Section IV), and point masses with continuous matter and shear (Section V). Section V also includes upper bounds on the number of fixed points.

II. FIXED POINTS DUE TO A GENERAL LENS

Some basic terminologies. Let $L = R^2 - A$, where A is a finite collection of points in R^2, and denote the Laplacian on R^2 by Δ. A *gravitational lens potential with infinite singularities* A is a smooth map $\psi : L \longrightarrow R$ such that $\Delta\psi(\mathrm{x}) \geq 0$ for all $\mathrm{x} \in L$, and $\psi(\mathrm{x}) \rightarrow -\infty$ as $\mathrm{x} \rightarrow a$ for each $a \in A$. We call ψ *isolated* if $\psi(\mathrm{x}) \rightarrow +\infty$ as $|\mathrm{x}| \rightarrow \infty$. Also, ψ is *nonsingular* when $A = \emptyset$; otherwise, ψ is *singular*. The (*single-plane*) *lensing map* induced by ψ is the map $\boldsymbol{\eta} : L \longrightarrow S$ defined by $\boldsymbol{\eta}(\mathrm{x}) = \mathrm{x} - \boldsymbol{\alpha}(\mathrm{x})$, where $S = R^2$ and $\boldsymbol{\alpha} = grad\,\psi$. We call L and S, resp., the *lens* and *light source plane* of $\boldsymbol{\eta}$, while $\boldsymbol{\alpha}$ is the *bending-angle vector field* of ψ.

A *lensed image* of a light source at $\mathrm{y} \in S$ is a solution x of the lens equation $\mathrm{y} = \boldsymbol{\eta}(\mathrm{x})$. If x_0 is a fixed point of $\boldsymbol{\eta}$, then x_0 is a lensed image of a light source at $\mathrm{y} = \mathrm{x}_0$. We can also assign a critical point type to lensed images. In fact, the lensed images of a light source at y are critical points of the *time-delay function* $T_{\mathrm{y}} : L \rightarrow R$ induced at y by ψ:

$$T_{\mathrm{y}}(\mathrm{x}) = \frac{|\mathrm{x} - \mathrm{y}|^2}{2} - \psi(\mathrm{x}).$$

We have $grad\,T_{\mathrm{y}}(\mathrm{x}) = 0$ if and only if $\boldsymbol{\eta}(\mathrm{x}) = \mathrm{y}$. The *magnification* of a lensed image x_0 of y is defined to be $1/|\det[\mathrm{Jac}\boldsymbol{\eta}](\mathrm{x}_0)| = 1/|\det[Hess\,T_{\mathrm{y}}](\mathrm{x}_0)|$. If the magnification of x_0 is 1 and a tiny disc is centered about y, then the resulting extended lensed image about x_0 has approximately the same area as the disc.

Morse Boundary Conditions B. It is practically unfeasible to study the fixed points of lensing maps due to arbitrary gravitational lens potentials ψ. However, if ψ satisfies certain "generic" properties, then we can obtain a lower bound and counting formula for the number of fixed points of the induced lensing map $\boldsymbol{\eta}$. First, we shall assume that ψ is nondegenerate. In fact, for almost all a in R^2 with $|a|$ sufficiently small, the linear perturbation,

$$\psi^a(\mathrm{x}) = \psi(\mathrm{x}) - a \cdot \mathrm{x},$$

of ψ is nondegenerate gravitational lens potential on L.[4] As $|a| \to 0$, the potential ψ^a becomes an arbitrarily small linear perturbation of ψ on any bounded open subset of L. Second, let D be a *regular domain* in L, i.e., a compact subset of L whose boundary ∂D is a disjoint union of smooth simple closed curves. Since the critical points of a nondegenerate ψ are isolated, we can choose D such that ∂D contains no critical point of ψ. *Denote by D_- the union of all open subsets of ∂D where the bending-angle vector field of ψ points into the interior of D.* If ψ is nondegenerate, then for almost all points $a \in R^2$, the critical points in D of a linear perturbation ψ^a tend uniformly and bijectively with preservation of critical point type, to those of ψ in D as $|a| \to 0$. Moreover, for $|a|$ sufficiently small, the restricted map $\psi^a|_{\partial D}$ is nondegenerate. Proofs of these results appear in Morse & Cairns.[5] We shall then define ψ to obey *Morse boundary conditions* B relative to a regular domain D in L if all critical points of ψ lying in D are nondegenerate, no critical point of ψ lies on ∂D, and the restriction $\psi|_{\partial D}$ is nondegenerate. This property is "generic" in the sense that it is satisfied by an arbitrarily small linear perturbation of ψ in any bounded open subset containing D.

Notation:

(1) $F(\boldsymbol{\eta}; D)$ = set of fixed points of $\boldsymbol{\eta}$ in D.

$|F(\boldsymbol{\eta}; D)|$ = total number of fixed points of $\boldsymbol{\eta}$ in D.

(2) $M_{min}(\psi; D)$ = total number of minima critical points of ψ in D.

$M_{sad}(\psi; D)$ = total number of saddle critical points of ψ in D.

Note that ψ has no maximum critical point (since $\Delta\psi(\mathrm{x}) \geq 0$ for all $\mathrm{x} \in L$).

(3) $M'_{min}(\psi|_{D_-})$ = total number of minima critical points of $\psi|_{D_-}$.

$M'_{max}(\psi|_{D_-})$ = total number of maxima critical points of $\psi|_{D_-}$.

Since a regular domain represents a portion of the sky where observations are made, it suffices to consider regular domains that are *closed g-hole discs* (i.e., a manifold that is diffeomorphic to a closed disc from which g disjoint open discs are excised). Think of the holes as places where a singularity of the gravitational lens is located.

Theorem 1 *Let $\boldsymbol{\eta} : L \to S$ be a lensing map induced by a gravitational lens potential ψ with infinite singularities A. Suppose that ψ obeys Morse boundary conditions B relative to a closed g-hole disc D, where each hole of D has exactly one element of A. [Note that the number of singularities of ψ may exceed g.] Then the fixed points of $\boldsymbol{\eta}$ in D satisfy:*

$$\begin{aligned} |F(\boldsymbol{\eta}; D)| &= 2M_{min}(\psi; D) \; + \; g \; - \; 1 \; + \; [M'_{min}(\psi|_{D_-}) - M'_{max}(\psi|_{D_-})] \\ &= 2M_{sad}(\psi; D) \; - \; g \; + \; 1 \; - \; [M'_{min}(\psi|_{D_-}) - M'_{max}(\psi|_{D_-})]. \end{aligned}$$

Proof: This theorem immediately follows from application of Morse inequalities B. [An introduction to these powerful inequalities, with detailed proofs, appears in Morse & Cairns[6].] First, since ψ is nondegenerate with no critical point on ∂D, compactness of D implies that ψ has finitely many critical points in D. Consequently, $|F(\boldsymbol{\eta}; D)| = M_{min}(\psi; D) \; + \; M_{sad}(\psi; D)$, which is finite. Second, because ψ also satisfies Morse boundary conditions B relative to D, Morse inequalities B yield:

$$\begin{aligned} M_{min}(\psi; D) &\geq B_0(D) - M'_{min}(\psi|_{D_-}) \\ M_{sad}(\psi; D) - M_{min}(\psi; D) &\geq [B_1(D) - M'_{max}(\psi|_{D_-})] - [B_0(D) - M'_{min}(\psi|_{D_-})] \\ M_{max}(\psi; D) - M_{sad}(\psi; D) + M_{min}(\psi; D) &= B_2(D) - [B_1(D) - M'_{max}(\psi|_{D_-})] \\ &\quad + [B_0(D) - M'_{min}(\psi|_{D_-})], \end{aligned}$$

where $B_i(D)$ is the i^{th} Betti number of D. The Betti numbers of a closed g-hole disc are: $B_0(D) = 1$, $B_1(D) = g$, and $B_2(D) = 0$ (e.g., Petters[7]). Since $M_{max}(\psi; D) = 0$ and $|F(\boldsymbol{\eta}; D)| = M_{min}(\psi; D) + M_{sad}(\psi; D)$, the theorem follows. **Q.E.D.**

In the sequel, we will usually abbreviate $F(\boldsymbol{\eta}; D)$ by $F(\boldsymbol{\eta})$, $M_{min}(\psi; D)$ by $M_{min}(\psi)$, etc. We will also define $\psi_- = -\psi$ and $\boldsymbol{\alpha}_- = -\boldsymbol{\alpha}$.

III. APPLICATIONS: NONSINGULAR ISOLATED LENSES

Theorem 2 *Let ψ be a nonsingular isolated nondegenerate gravitational lens potential. Suppose that ψ obeys Morse boundary conditions B. Then fixed points of the lensing map $\boldsymbol{\eta}$ induced by ψ exist, and satisfy the following:*

(1) $1 \leq |F(\boldsymbol{\eta})| < \infty$.

(2) Odd-Number Fixed-Point Theorem: $|F(\boldsymbol{\eta})| = 2M_{min}(\psi) \; - 1 = 2M_{sad}(\psi) \; + \; 1$.

Proof: The fixed points of $\boldsymbol{\eta}$ coincide with the critical points of ψ_-. Since ψ is isolated, all critical points of ψ_- lie in the interior of some closed disc D. By choosing the radius of D sufficiently large, we can also assume that $\boldsymbol{\alpha}_-$ is inward pointing on ∂D (so $\boldsymbol{\alpha}$ is outward pointing on ∂D). Consequently, $|F(\boldsymbol{\eta})| < \infty$ and D_- is empty. The theorem is now a consequence of Theorem 1. **Q.E.D.**

Without explicitly knowing the gravitational lens potential modeling a particular smooth nonsingular isolated mass distribution, Theorem 2 enables us to conclude that the lens produces an *odd* number of fixed points and, hence, at least one.

Example (elliptic potential): An elliptical potential with ellipticity ϵ and finite core x_c is of the form:

$$\psi_{ep}(\mathrm{x}) = C\sqrt{x_{cr}^2 + (1-\epsilon)\mathrm{u}^2 + (1+\epsilon)\mathrm{v}^2},$$

where $\mathrm{x} = (\mathrm{u}, \mathrm{v})$, $C > 0$, $x_{cr} > 0$, and $0 \leq \epsilon < 1$. The bending-angle vector field is:

$$\boldsymbol{\alpha}_{ep}(\mathrm{x}) = C^2 \left((1-\epsilon)\mathrm{u}, (1+\epsilon)\mathrm{v}\right) / \psi_{ep}(\mathrm{x}).$$

There is only one fixed point, namely, the origin.

Although Theorem 2 is stated for isolated potentials, the proof of the theorem shows that we can also determine an Odd-Number Fixed-Point Theorem for any nonsingular potential for which $M'_{min}(\psi|_{D_-}) - M'_{max}(\psi|_{D_-})$ is an even number.

Example (perturbed Plummer potential): A perturbed Plummer potential with shear γ at infinity is of the form

$$\psi_{ppl}(\mathrm{x}) = \frac{\kappa_0}{2} \ln \left(1 + \mid \mathrm{x} \mid^2\right) - \frac{\gamma}{2}(\mathrm{u}^2 - \mathrm{v}^2),$$

where κ_0 and γ are positive. This potential is not isolated. Employing polar coordinates (R, θ), the potential is approximated for large values of R by $\tilde{\psi}(R, \theta) = -\gamma R^2 \cos 2\theta$. On a large disk of radius R, D_- is approximated by the intervals $-\pi/4 < \theta < -\pi/4$ and $3\pi/4 < \theta < 5\pi/4$. Hence $\tilde{\psi}|_{D_-}$ has two critical points, both minima. Hence $M'_{min}(\psi|_{D_-}) - M'_{max}(\psi|_{D_-}) = 2 - 0 = 2$, and so there will be an odd number of fixed points.

In fact, for this potential we can explicitly compute the fixed points. Since the associated bending-angle vector field is

$$\boldsymbol{\alpha}_{ppl}(\mathrm{x}) = \left(\left(\frac{\kappa_0}{1 + \mid \mathrm{x} \mid^2} - \gamma \right) \mathrm{u}, \left(\frac{\kappa_0}{1 + \mid \mathrm{x} \mid^2} + \gamma \right) \mathrm{v} \right),$$

there is always a real fixed point at $\mathrm{x} = (0, 0)$. If $\kappa_0 > \gamma$, two more fixed points occur at $\mathrm{x} = \left(\pm\sqrt{\frac{\kappa_0}{\gamma} - 1}, 0\right)$. Note that for $\kappa_0 < \gamma$, the origin is a saddle; for $\kappa_0 > \gamma$, the origin is a minimum, and the other two fixed points are saddle points. Figure 1 depicts the locations of these fixed points. Notice the lensing of a disc light source centered at the fixed point in Figure 2.

In the (typical) case that no explicit formula for the location of the fixed points exists, iterative methods may be used to numerically approximate the fixed points (e.g., Press *et al.*[8]) These methods typically use some sort of Newton-iteration and require the use of an initial guess (or "seed") to start the iteration; these methods are not guaranteed to find all of the fixed points that exist, even if the algorithm is run using thousands of initial seeds. It is imperative, therefore, to have estimates and bounds on the number and location of fixed points, so that we may be more confident in the results of our numerics. All of the theorems in this paper provide estimates of this sort, and therefore provide important practical information to computational mathematicians and physicists.

The reader may wonder about the likelihood of a fixed point located on a critical curve. These situations appear in Figures 1 and 2, but, in general, these are not generic occurrences. If ξ is a fixed point, then $grad\,\psi(\xi) = 0$, but this condition is independent of the condition that ξ is on a critical curve: $\det[d\boldsymbol{\eta}](\xi) = \det[I - Hess(\psi)](\xi) = 0$. These two conditions mean that the condition that a fixed point is on a critical curve is a codimension-1 phenomenon. For example, evaluating $\det[I - Hess(\psi)]$ for the perturbed Plummer model at the fixed point gives a determinant of $(1 + 2\gamma(1 - \gamma/\kappa_0))\,(1 - 2\gamma)$ and this quantity only vanishes along the curve $\kappa_0 = 2\gamma^2/(1 + 2\gamma)$ in the two-dimensional parameter space.

It is even more unlikely that a fixed point on a critical curve is mapped onto a cusp point by $\boldsymbol{\eta}$. The point ξ is a cusp point if the critical curve is nonsingular at ξ, and the tangent vector to the critical curve is contained in the kernel of $d\boldsymbol{\eta}$. In symbols, if T is a tangent vector to the critical curve at ξ, then $d\boldsymbol{\eta}(\xi) \cdot T = 0$. (To be certain that ξ is mapped to a "mathematical cusp," and not a higher-order singularity, we should also check that the family of tangent vectors near ξ pass transversally through the kernel of $d\boldsymbol{\eta}$, but we will not go into these details here.) For example, since the Plummer potential is invariant under the Z_2-symmetry $(u, v) \to (u, -v)$, we know that critical curves passing through the line $v = 0$ must have vertical tangent vectors. The Jacobian of the lensing map (evaluated at the fixed point) sends the vector $[0, 1]$ to the vector $[0, 1 - 2\gamma]$, and so we see that $d\boldsymbol{\eta}(\xi)$ sends a tangent vector to the zero vector only when $\gamma = 1/2$. Of course, for extended light sources, there is an open set of parameter values for which the geometry of Figure 2 would be observable, so the physical realization of this figure will not be as rare as is indicated by a mathematical analysis of the point-source case.

IV. APPLICATIONS: POINT MASSES WITH CONTINUOUS MATTER AND SHEAR

A. Preliminaries

The negative of the gravitational lens potential due g point masses m_ℓ at positions $\boldsymbol{\xi}_\ell$ on a plane lens with constant-density continuous matter κ_c (in units of the critical density) and non-normalized shear $\gamma \geq 0$ (at infinity) is given by:

$$\psi_-(\mathrm{x}) = \frac{(\gamma - \kappa_c)}{2}\mathrm{u}^2 - \frac{(\gamma + \kappa_c)}{2}\mathrm{v}^2 - \sum_{\ell=1}^{g} m_\ell \ln |\mathrm{x} - \boldsymbol{\xi}_\ell|, \tag{1}$$

where $\mathrm{x} = (\mathrm{u}, \mathrm{v})$. The fixed points due to ψ are solutions x of the equation $\boldsymbol{\alpha}_-(\mathrm{x}) = 0$, where

$$\boldsymbol{\alpha}_-(\mathrm{x}) = ((-\kappa_c + \gamma)\mathrm{u},\ (-\kappa_c - \gamma)\mathrm{v}) - \sum_{\ell=1}^{g} m_\ell \frac{\mathrm{x} - \boldsymbol{\xi}_\ell}{|\mathrm{x} - \boldsymbol{\xi}_\ell|^2}. \tag{2}$$

The lensing map $\boldsymbol{\eta} : L \to S$ induced by ψ is given by $\boldsymbol{\eta}(\mathrm{x}) = \mathrm{x} + \boldsymbol{\alpha}_-(\mathrm{x})$, where $L = R^2 - \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_g\}$ and $S = R^2$. Explicitly,

$$\boldsymbol{\eta}(\mathrm{x}) = \begin{pmatrix} 1 - \kappa_c + \gamma & 0 \\ 0 & 1 - \kappa_c - \gamma \end{pmatrix} \mathrm{x} - \sum_{\ell=1}^{g} m_\ell \frac{\mathrm{x} - \boldsymbol{\xi}_\ell}{|\mathrm{x} - \boldsymbol{\xi}_\ell|^2}. \tag{3}$$

Now, since $\psi_-(\mathrm{x}) \to +\infty$ as $\mathrm{x} \to \boldsymbol{\xi}_\ell$, where $\ell = 1, \ldots, g$, there is an open disc about $\boldsymbol{\xi}_\ell$ that contains no critical point of ψ_-. In addition, if $\kappa_c = \gamma$, then $\psi_-(\mathrm{x}) \to -\infty$ as $|\mathrm{x}| \to \infty$. Moreover, if $\kappa_c \neq \gamma$, then $|\boldsymbol{\alpha}_-(\mathrm{x})| \to +\infty$ as $|\mathrm{x}| \to \infty$. Consequently, all critical points lie in a bounded region. Let D be a closed g-hole disc, where each hole of D has exactly one singularity of ψ. By choosing the circumradius of the inner-boundary curves and inradius of the outer-boundary curve of D sufficiently small and large, resp., we can assume that all critical points of ψ_- lie in the interior of D. Hence, *if ψ_- is nondegenerate, then it has finitely many critical points, that is, $|F(\boldsymbol{\eta})| = M_{min}(\psi) + M_{sad}(\psi) < \infty$, where $\boldsymbol{\eta}$ is the lensing map induced by ψ.* Note that $M_{max} = 0$ since $\Delta\psi = 2\kappa_c \geq 0$.

Though Theorem 1 yields counting formulas and lowers bounds on $|F(\boldsymbol{\eta})|$, it provides no upper bound. However, by introducing complex variables (Bourassa, Kantowski, and Norton[9]; Witt[10]), we shall determine upper bounds on $|F(\boldsymbol{\eta})|$ for point masses on a lens plane with shear and continuous matter.

Express x and $\boldsymbol{\xi}_\ell$ in complex form: $\mathrm{x} = z = \mathrm{u} + i\mathrm{v}$ and $\boldsymbol{\xi}_\ell = \xi_\ell^1 + i\xi_\ell^2$. By Eq.(2), we must investigate the zeros of

$$\boldsymbol{\alpha}_-(z,\bar{z}) = -\kappa_c z + \gamma\bar{z} + \sum_{\ell=1}^{g} \frac{m_\ell}{\bar{\boldsymbol{\xi}}_\ell - \bar{z}}, \tag{4}$$

which is equivalent to finding the simultaneous roots of

$$p(z,\bar{z}) = -\kappa_c z \prod_{\ell=1}^{g} (\bar{\boldsymbol{\xi}}_\ell - \bar{z}) + \gamma\bar{z} \prod_{\ell=1}^{g} (\bar{\boldsymbol{\xi}}_\ell - \bar{z}) + \sum_{\ell=1}^{g} m_\ell \prod_{\substack{r=1\\ r\neq\ell}}^{g} (\bar{\boldsymbol{\xi}}_r - \bar{z}) \tag{5}$$

and

$$\bar{p}(z,\bar{z}) = \gamma z \prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z) - \kappa_c\bar{z} \prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z) + \sum_{\ell=1}^{g} m_\ell \prod_{\substack{r=1\\ r\neq\ell}}^{g} (\boldsymbol{\xi}_r - z). \tag{6}$$

Thus, if x is a fixed point of $\boldsymbol{\eta}$, then $(z,\bar{z})$ is a root of both (5) and (6). We can then translate the study of fixed points into that of the common zeros of $p(z,\bar{z})$ and $\bar{p}(z,\bar{z})$. Applications of the theory of resultants will then yield upper bounds on $|F(\boldsymbol{\eta})|$.

B. Fixed Points due to Point Masses with $\kappa_c = 0$ and $\gamma \geq 0$

Throughout this subsection, assume that $\gamma \geq \kappa_c = 0$.

Notation:

(1) $M_{tot} = m_1 + \cdots + m_g$.

(2) $\hat{M}_\ell = M_{tot} - m_\ell$.

Theorem 3 *Let ψ be a nondegenerate gravitational lens potential due to g point masses on a lens plane with shear $\gamma \geq 0$ and no continuous matter ($\kappa_c = 0$). Then the fixed points of the lensing map $\boldsymbol{\eta}$ induced by ψ satisfy the following:*

(1) The set $F(\boldsymbol{\eta})$ consists of all roots of the equation $\bar{p}(z) = 0$, where

$$\bar{p}(z) = \gamma z \prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z) + \sum_{\ell=1}^{g} m_\ell \prod_{\substack{r=1 \\ r\neq\ell}}^{g} (\boldsymbol{\xi}_r - z) \tag{7}$$

and has only simple zeros. Consequently,

$$|F(\boldsymbol{\eta})| = M_{sad}(\psi) = \begin{cases} g+1 & \text{if } \ \gamma > 0 \\ g-1 & \text{if } \ \gamma = 0. \end{cases} \tag{8}$$

(2) The fixed points $y_1, \ldots, y_{n(\gamma)}$ of $\boldsymbol{\eta}$ obey:

$$y_1 + \cdots + y_{n(\gamma)} = \begin{cases} \frac{\boldsymbol{\xi}_1 + \cdots + \boldsymbol{\xi}_g}{\gamma} & \text{if } \ \gamma > 0 \\ \\ \frac{\hat{M}_1 \boldsymbol{\xi}_1 + \cdots + \hat{M}_g \boldsymbol{\xi}_g}{M_{tot}} & \text{if } \ \gamma = 0, \end{cases} \tag{9}$$

where $n(\gamma) = g+1$ if $\gamma > 0$, and $n(\gamma) = g-1$ if $\gamma = 0$. Moreover, no fixed point of $\boldsymbol{\eta}$ has magnification 1.

Proof:

(1) We have $|F(\boldsymbol{\eta})| = M_{sad}(\psi) \ + \ M_{min}(\psi)$. But $\Delta\psi = 2\kappa_c = 0$. Hence, $M_{min}(\psi) = 0$. Since $\kappa_c = 0$, equations (5) and (6) are equivalent to the single-variable equation $\bar{p}(z) = 0$, where $\bar{p}$ is given by Eq.(7). In addition,

$$\frac{d\bar{\boldsymbol{\alpha}}_-}{dz}(z) = \frac{p'(z) \prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z) - p(z) \left[\prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z)\right]'}{\prod_{\ell=1}^{g} (\boldsymbol{\xi}_\ell - z)^2}$$

and $\det[Hess\,\psi](\mathrm{y}) = |\,(d\bar{\boldsymbol{\alpha}}_-/dz)\,(\mathrm{y})|^2 \neq 0$, where $\bar{\boldsymbol{\alpha}}_-$ is the conjugate of $\boldsymbol{\alpha}_-$ and $'$ denotes differentiation relative to z. Hence, all zeros of $p(z)$ are simple. The remainder of part (1) now follows from the degree of $\bar{p}(z)$.

(2) We want to expand the products in Equation (6). Recall that the kth elementary symmetric function in $t_1, \ldots, t_n$ is

$$s_k = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \boldsymbol{\xi}^{\ell}_{j_1} \cdots \boldsymbol{\xi}^{\ell}_{j_k},$$

where $1 \leq k \leq n$ (e.g., $s_1 = t_1 + \cdots + t_n$ and $s_n = t_1 \cdots t_n$). Let

$$\boldsymbol{\xi}^{\ell}_j = \begin{cases} \boldsymbol{\xi}_j & j < \ell \\ \boldsymbol{\xi}_{j+1} & j \geq \ell \end{cases},$$

where $1 \leq j \leq g-1$ and $1 \leq \ell \leq g$. Denote the kth elementary symmetric function in $\boldsymbol{\xi}^{\ell}_1, \ldots, \boldsymbol{\xi}^{\ell}_{g-1}$ by $\boldsymbol{\xi}^{\ell}_k$ (e.g., $\boldsymbol{\xi}^1_j = \boldsymbol{\xi}_{j+1}$ and $\boldsymbol{\xi}^g_j = \boldsymbol{\xi}_j$). Note that $s^{\ell}_1 = \boldsymbol{\xi}^{\ell}_1 + \cdots + \boldsymbol{\xi}^{\ell}_{g-1} = \boldsymbol{\xi}_1 + \cdots + \hat{\boldsymbol{\xi}}_{\ell} + \cdots + \boldsymbol{\xi}^{\ell}_g$, where $\hat{\boldsymbol{\xi}}_{\ell}$ is omitted, and $s^{\ell}_{g-1} = \boldsymbol{\xi}^{\ell}_1 \cdots \boldsymbol{\xi}^{\ell}_{g-1}$. Also, $s^1_1 = \boldsymbol{\xi}_2 + \cdots + \boldsymbol{\xi}_{g-1}$ and $s^g_{g-1} = \boldsymbol{\xi}_1 \cdots \boldsymbol{\xi}_{g-1}$.

We expand Equation (7) as follows:

$$\begin{aligned}
\bar{p}(z) &= \gamma z \prod_{\ell=1}^{g} (\boldsymbol{\xi}_{\ell} - z) + (-1)^{g-1} \sum_{\ell=1}^{g} m_{\ell} \prod_{j=1}^{g-1} (z - \boldsymbol{\xi}^{\ell}_j) \\
&= (-1)^g z \left(z^g - s_1 z^{g-1} + \cdots + (-1)^{g-1} s_g \right) \\
&\quad + (-1)^{g-1} \left\{ M_{tot} z^{g-1} - \sum_{\ell=1}^{g} m_{\ell} (\boldsymbol{\xi}_1 + \cdots + \hat{\boldsymbol{\xi}}_{\ell} + \cdots + \boldsymbol{\xi}_g) z^{g-2} + \text{lower-order terms} \right\} \\
&= (-1)^g \left(z^{g+1} - s_1 z^g + \cdots + (-1)^{g-1} s_g z \right) \\
&\quad + (-1)^{g-1} \left\{ M_{tot} z^{g-1} - \left(\sum_{\ell=1}^{g} \hat{M}_{\ell} \boldsymbol{\xi}_{\ell} \right) z^{g-2} + \text{lower-order terms} \right\}.
\end{aligned}$$

Since the sum of the zeros of a complex polynomial $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is $-a_{n-1}/a_n$, the first claim in part (2) of the Theorem follows. For the second claim, observe that the determinant of the Jacobian of $\boldsymbol{\eta}$ is given by

$$\det J(z, \bar{z}) = \frac{\partial w}{\partial z} \overline{\frac{\partial w}{\partial z}} - \frac{\partial w}{\partial \bar{z}} \overline{\frac{\partial w}{\partial \bar{z}}} = 1 - \left| \frac{d\bar{\boldsymbol{\alpha}}_-}{dz}(z) \right|^2.$$

Hence, a fixed-point lensed image z_0 has magnification 1 if and only if $|(d\bar{\boldsymbol{\alpha}}_-/dz)(z_0)| = 0$. Nondegeneracy of ψ rules out this possiblity. **Q.E.D.**

Theorem 3 shows that a one point-mass lens has no fixed points, while a point-mass with shear generically produces two fixed points, as shown in Figures 3 and 4. In the nongeneric case (i.e., the gravitational lens potential is degenerate), then latter has one fixed point of multiplicity two; it is a minimum lensed image with magnification 1. A nondegenerate binary lens has a fixed point at position

$$\mathrm{y} = \frac{m_1 \boldsymbol{\xi}_2 + m_2 \boldsymbol{\xi}_1}{m_1 + m_2}.$$

If the positions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ of the masses m_1 and m_2 are switched, then y is the *center of mass* of the resulting system. For a ternary lens system, the fixed points can also be located theoretically. In fact, note that zeros of Eq. (7) are solutions of

$$\sum_{\ell=1}^{g} \frac{m_\ell}{z - \boldsymbol{\xi}_\ell} = 0.$$

The zeros of this equation for $g = 3$ was studied by Siebeck as early as 1864. He established (see Marden[11]) that the zeros are the foci of the ellipse which meets the line segment between $\boldsymbol{\xi}_j$ and $\boldsymbol{\xi}_k$ at the point ζ_{jk}, where $|\zeta_{jk} - \boldsymbol{\xi}_j| = (m_j/m_k)|\zeta_{jk} - \boldsymbol{\xi}_k|$ (see Figure 3). Siebeck also covered the case for general g, where the fixed points become the foci of a certain algebraic curve. Part (2) of Theorem 3 further shows that the sum of the positions of the fixed points (counting with multiplicity) of a g-point mass lens, where $g \geq 2$, is the center of mass of the lens system if m_ℓ is replaced by $\hat{M}_\ell$.

C. Fixed Points due to Point Masses with $0 \leq \gamma \leq \kappa_c$ and $0 < \kappa_c < \gamma$

1. Counting Formulas: $0 \leq \gamma \leq \kappa_c$ and $0 < \kappa_c < \gamma$

For $0 \leq \gamma < \kappa_c$, we define the ellipse parametrized by

$$\theta \to \left(\sqrt{\frac{2}{\kappa_c - \gamma}}\, R\cos(\theta), \sqrt{\frac{2}{\kappa_c + \gamma}}\, R\sin(\theta)\right),$$

whereas we choose

$$\theta \to \left(\sqrt{\frac{2}{\gamma - \kappa_c}}\, R\cos(\theta), \sqrt{\frac{2}{\kappa_c + \gamma}}\, R\sin(\theta)\right)$$

when $0 \leq \kappa_c < \gamma$. Let $D(R)$ be the interior of the ellipse punctured by disjoint open discs $D_{\boldsymbol{\xi}_1}, \ldots, D_{\boldsymbol{\xi}_g}$ centered at the point masses. By choosing the radii of the discs $D_{\boldsymbol{\xi}_\ell}$ sufficiently small, we can assume that at points of the boundary $\partial D_{\boldsymbol{\xi}_\ell}$, $grad\, \psi_-$ points into the exterior of $D(R)$. Moreover, for R sufficiently large we can also assume that all critical points of ψ_- lie in $D(R)$. *Henceforth, we shall assume the previous properties hold for* $D(R)$ *and all critical points of* ψ_- *lie in the interior of* $D(R)$. Finally, let $D_-(R)$ be the union of all submanifolds of $\partial D(R)$ at whose points $grad\, \psi_-$ points into the *exterior* of $D(R)$. In other words, $D_-(R) = D_-$.

Theorem 4 *Suppose that* ψ *obeys Morse boundary conditions* B *relative to* $D(R)$.

(1) If $0 \leq \gamma \leq \kappa_c$, *then* $|F(\boldsymbol{\eta})| = 2M_{min}(\psi) + g - 1 = 2M_{sad}(\psi) - g + 1$.

(2) If $0 < \kappa_c < \gamma$, *then* $|F(\boldsymbol{\eta})| = 2M_{min}(\psi) + g + 1 = 2M_{sad}(\psi) - g - 1$.

The theorem shows that $|F(\boldsymbol{\eta})|$ and g have opposite parity (i.e., even or oddness).

Proof: We have $M_{min}(\psi) = M_{max}(\psi_-)$, $M_{sad}(\psi) = M_{sad}(\psi_-)$, $M'_{min}(\psi|_{D_-(R)}) = M'_{max}(\psi_-|_{D_-(R)})$, and $M'_{max}(\psi|_{D_-(R)}) = M'_{min}(\psi_-|_{D_-(R)})$. Theorem 1 implies that

$$\begin{aligned}
|F(\boldsymbol{\eta})| &= 2M_{min}(\psi) + g - 1 + [M'_{max}(\psi_-|_{D_-(R)}) - M'_{min}(\psi_-|_{D_-(R)})] \\
&= 2M_{sad}(\psi) - g + 1 - [M'_{max}(\psi_-|_{D_-(R)}) - M'_{min}(\psi_-|_{D_-(R)})] \qquad (10)
\end{aligned}$$

(1) For R sufficiently large, the hypothesis that $\gamma \leq \kappa_c$ implies that $\boldsymbol{\alpha}_-$ points into the interior of $D(R)$ at all points of the outer boundary of $\partial D(R)$. Recall that if $\gamma = \kappa_c$, then

$\psi_-(\mathrm{x}) \to -\infty$ as $|\mathrm{x}| \to \infty$. In addition, $\boldsymbol{\alpha}_-$ points into the exterior of $D(R)$ at all points of the g inner-boundary curves of $\partial D(R)$. Consequently, $D_-(R)$ is a union of g disjoint circles S^1. Since $\psi_-|_{D_-(R)}$ is nondegenerate, the Morse inequalities for boundaryless manifolds (see Milnor[12]) also apply and yield

$$\begin{aligned} M'_{max}(\psi_-|_{D_-(R)}) - M'_{min}(\psi_-|_{D_-(R)}) &= B_1(D_-(R)) - B_0(D_-(R)) \\ &= g(B_1(S^1) - B_0(S^1)) = g(1-1) = 0. \end{aligned}$$

Application of Eq. (10) yields the desired result.

(2) By restricting ψ_- to ∂D and employing our parametrization of the ellipse, we obtain

$$\psi_-|_{\partial D(R)}(\theta) = R^2\cos(2\theta) - \sum_{\ell=1}^{g} m_\ell \ln\left(R\sqrt{\tilde{G}_\ell(\theta)}\right)$$

where

$$\tilde{G}_\ell(\theta) = \frac{2}{\gamma-\kappa_c}\cos^2\theta + \frac{2}{\kappa_c+\gamma}\sin^2\theta - \frac{2\boldsymbol{\xi}_\ell^1}{R}\sqrt{\frac{2}{\gamma-\kappa_c}}\cos\theta - \frac{2\boldsymbol{\xi}_\ell^2}{R}\sqrt{\frac{2}{\kappa_c+\gamma}}\sin\theta + \left(\frac{|\,\boldsymbol{\xi}_\ell\,|}{R}\right)^2.$$

For large values of R, the function $\psi_-|_{\partial D(R)}(\theta)$ is approximately $R^2\cos(2\theta)$. Hence, $D_-(R)$ is approximated by portions of ∂D for $-\pi/4 < \theta < \pi/4$. and $3\pi/4 < \theta < 5\pi/4$. Thus the map $\psi_-|_{D_-(R)}(\theta)$ has two critical points, both maxima: $M'_{max}(\psi_-|_{D_-(R)}) = 2$ and $M'_{min}(\psi_-|_{D_-(R)}) = 0$. The result now follows from Eq. (10). **Q.E.D.**

2. Bounds: $0 \le \gamma \le \kappa_c$ and $0 < \kappa_c < \gamma$

Theorem 5 *Suppose that ψ obeys Morse boundary conditions B.*

(1) If $0 \le \gamma \le \kappa_c$, then:

$$g-1 \le |F(\boldsymbol{\eta})| \le \begin{cases} (g+1)^2 & \text{if } \gamma > 0 \\ g^2-1 & \text{if } \gamma = 0. \end{cases}$$

(2) If $0 < \kappa_c < \gamma$, then: $g+1 \le |F(\boldsymbol{\eta})| \le (g+1)^2$.

Proof:

For the lower bounds in the theorem, note that Theorem 4(1) implies that if $0 \leq \gamma \leq \kappa_c$, then $|F(\boldsymbol{\eta})| = 2M_{min}(\psi) + g - 1 \geq g - 1$. In addition, by Theorem 4(2), we have $|F(\boldsymbol{\eta})| = 2M_{min}(\psi) + g + 1 \geq g + 1$ for $0 < \kappa_c < \gamma$.

In order to prove the upper bounds in the theorem, we must introduce some facts from the theory of resultants. The x_1-Sylvester resultant of two polynomials

$$P(x_1, x_2) = a_0(x_2) + a_1(x_2)x_1 + \ldots + a_m(x_2)x_1^m$$

and

$$Q(x_1, x_2) = b_0(x_2) + b_1(x_2)x_1 + \ldots + b_n(x_2)x_1^n,$$

is defined by

$$\mathrm{Res}(x_2) = \det\left(S_{m+n}\right),$$

where

$$S_{m+n} = \begin{bmatrix} a_m & a_{m-1} & a_{m-2} & \cdot & \cdot & \cdot & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdot & \cdot & \cdot & a_1 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & a_{m-2} & \cdot & \cdot & \cdot & a_1 & a_0 \\ b_n & b_{n-1} & \cdot & \cdot & \cdot & b_0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_n & b_{n-1} & \cdot & \cdot & \cdot & b_0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & b_n & b_{n-1} & \cdot & \cdot & \cdot & b_0 \end{bmatrix}. \tag{11}$$

Explicitly, the ℓth-row of S_{m+n}, where $1 \leq \ell \leq n$, consists of $\ell - 1$ zeros, followed by $a_m, a_{m-1}, \ldots, a_0$, and the remainder of the row filled with zeros; for $n + 1 \leq \ell \leq m + n$, the ℓth–row has $\ell - n - 1$ zeros followed by $b_n, b_{n-1}, \ldots, b_0$, and the remaining entries are zeros.

The Resultant Theorem (see Abhyankar[13,14]) states that β satisfies $\mathrm{Res}(\beta) = 0$ if and only if either $a_m(\beta) = b_n(\beta) = 0$, or, $P(\alpha, \beta) = Q(\alpha, \beta) = 0$ for some α. Another fact we shall need is that the resultant is a polynomial of degree n in the coefficients a_ℓ and of degree m in b_ℓ. That is, if we write $\mathrm{Res}(x_2) = \mathrm{Res}(a_0, \ldots, a_m, b_0, \ldots, b_n)$, then

$$\mathrm{Res}(a_0, \ldots, a_m, b_0, \ldots, b_n) = \sum S_{i_0 \ldots i_m j_0 \ldots j_n} a_0^{i_0} \ldots a_m^{i_m} b_0^{j_0} \ldots b_n^{j_n}$$

and $i_0 + \ldots + i_m = n$ and $j_0 + \ldots + j_n = m$. Therefore,

$$\deg[\mathrm{Res}(x_2)] \leq (n) \left(\max \left\{\deg[a_\ell]_{\ell=0}^m\right\}\right) + (m) \left(\max \left\{\deg[b_\ell]_{\ell=0}^n\right\}\right). \tag{12}$$

The previous results will now be applied to Eqs. (5) and (6) to obtain the upper bounds in theorem. We begin with $\gamma = 0$. We then consider the common roots of

$$p(z, \bar{z}) = a_1(\bar{z})z + a_0(\bar{z})$$

and

$$\bar{p}(z, \bar{z}) = b_g(\bar{z})z^g + \cdots + b_0(\bar{z}).$$

Denote the z-resulant of p and $\bar{p}$ by $\mathrm{Res}(\bar{z})$. If a is a fixed point, then $(a, \bar{a})$ is a common zero of p and $\bar{p}$. By the Resultant Theorem, a is a solution of $\mathrm{Res}(\bar{z}) = 0$. Since $\gamma = 0$,

$$\max \left\{\deg[a_0], \deg[a_1]\right\} = g$$

and

$$\max \left\{\deg[b_\ell]_{\ell=0}^g\right\} = 1,$$

Eq. (12) implies

$$|F(\boldsymbol{\eta})| \leq \deg[\mathrm{Res}(\bar{z})] \leq (g)(g) + (1)(1) = g^2 + 1.$$

Expanding the coefficients $a_1(\bar{z})$ and $b_g(\bar{z})$ we find that

$$\begin{aligned} a_1(\bar{z}) &= -\kappa_c \prod_{\ell=1}^{g} (\bar{\boldsymbol{\xi}}_\ell - \bar{z}) \\ &= (-1)^{g+1} \kappa_c \left(\bar{z}^g - \bar{s}_1 \mathrm{x}^{g-1} + \cdots + (-1)^{g-2} \bar{s}_{g-1} \bar{z} + (-1)^{g-1} \bar{s}_g\right) \end{aligned}$$

and

$$b_g(\bar{z}) = (-1)^{g+1}\kappa_c \bar{z}.$$

Without loss of generality, assume that $\boldsymbol{\xi}_1 = 0$. Then $\bar{z} = 0$ is a common zero of $a_1(\bar{z})$ and $b_g(\bar{z})$. By the Resultant Theorem, $\bar{z} = 0$ is a zero of $\mathrm{Res}(\bar{z})$. Since the origin is the position of the first point mass, $\bar{z} = 0$ is not a fixed point; hence, $|F(\boldsymbol{\eta})| \leq g^2$. By Theorem 4(1), $|F(\boldsymbol{\eta})|$ is even if and only if g is odd. Thus, we can reduce the upper bound by one: $|F(\boldsymbol{\eta})| \leq g^2 - 1$.

The proof for the upper bound when $\gamma > 0$ and $\kappa_c > 0$ (either $\gamma \leq \kappa_c$ or $\gamma > \kappa_c$ may apply) follows the proof above, except that Equation (5) now has coefficients (in $\bar{z}$) of maximal degree $g+1$ and Equation (6) is now a degree $g+1$ polynomial in z. Hence, by Equation (12),

$$|F(\boldsymbol{\eta})| \leq (g+1)(g+1) + (1)(1) = g^2 + 2g + 2.$$

Again we can reduce this bound by one by noting that $\bar{z} = 0$ is a common root of $a_1(\bar{z})$ and $b_g(\bar{z})$, but the parity argument no longer applies. Thus we may only claim that $|F(\boldsymbol{\eta})| \leq (g+1)^2$. **Q.E.D.**

As a last example, we again consider the ternary point-mass system with continuous matter and shear. Theorem 4 predicts that there will be an even number of fixed points when $g = 3$. Theorem 5 predicts that $2 \leq |F(\boldsymbol{\eta})| \leq 8$ if $0 = \gamma < \kappa_c$. If we perturb the lensing system shown in Figure 3 to, say, $\gamma = 0$ and $\kappa_c = 0.1$, then we find that the constellation is essentially the same, but that the two fixed points slightly perturb to the locations $(0.92698, 0.52446)$ and $(-0.28041, 0.14101)$. By contrast, when $0 < \kappa_c < \gamma$, Theorem 5 predicts that $4 \leq |F(\boldsymbol{\eta})| \leq 16$. If we perturb the lensing system shown in Figure 3 to $\kappa_c = 0.1$ and $\gamma = 0.3$, then we find that two new fixed points appear "from infinity," resulting in four fixed points: $(0.96570, 0.54376)$, $(-0.246848, 0.13689)$, $(-3.81466, -100.95612)$, and $(4.0949, 56.2398)$.

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Figure 1: A superposition of the critical curves (grey) and caustics (black) for the perturbed Plummer model, $\kappa_0 = 0.75, \gamma = 0.5$. There are three fixed points (marked by boxes), one at the origin and one at each cusp $(\pm 1/\sqrt{2}, 0)$.

Figure 2: (Upper) An extended light source is placed at each fixed point, A, B, and C, in the perturbed Plummer model. (Lower) Images of the extended sources. The fixed point at the origin has three images; the fixed point at each cusp is arc-like.

Figure 3: A superposition of the critical curves (grey) and caustics (black) for three equal masses, $m_i = 1$, $\kappa_c = \gamma = 0$. The positions of the masses are indicated by dots ($\bullet$): $d_1 = (-1, 0), d_2 = (1, 0), d_3 = (1, 1)$. The position of the fixed points are marked by boxes ($\square$): (0.93912, 0.51675) and (-0.27245, 0.14992). The insert illustrates that the fixed points are foci of an ellipse inscribed in the triangle defined by the three masses and meeting the triangle at the pair-wise centers of the sides.

Figure 4: (Upper) An extended light source located at one of the fixed points, that crosses a caustic. (Lower) Four images of the source. The arc-like image is due to the fact that the source crosses a caustic.