How to Lose at Tetris

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Introduction

Tetris is a computer game which has obsessed many computer users and attracted much attention, despite the simplicity of its rules. This paper addresses the question: "can you 'win' the game Tetris?"

Designed by Soviet mathematician Alexey Pazhitnov in the late eighties and imported to the United States by Spectrum Holobyte, Tetris won a record number of software awards in 1989 [4]. Versions of Tetris are sold for most personal computers. There are Tetris arcade games, Tetris Nintendo cartridges, and hand-held Tetris games; Tetris has been played on machines ranging from mainframes to calculators. The game's success has prompted the invention of several similar games, including Hextris, Welltris, and Wordtris.

Although mathematicians have spent many hours "studying" Tetris, surprisingly little is known about the mathematical properties of the game. Much research has been done on the subject of covering rectangles with sets of polyominoes [2,3,5,6]; Tetris adds a new twist to this familiar problem.

The game takes place on a grid or "board" ten units wide and twenty units tall. When the game starts, the board is empty. Then tetrominoes, groups of four connected "cells", each cell covering exactly one grid square, appear at the top of the board and fall row by row toward the bottom of the board (see Figure 1). When a tetromino reaches the bottom or a point where it can fall no further without two or more cells overlapping, it remains in that spot and another tetromino (randomly selected from the set of seven possible tetrominoes) appears at the top of the board. The player uses rotations and horizontal translations to orient the tetrominoes as they fall, attempting to cover rows of the board with cells. When a row is covered, the cells on that row are removed from the board and the cells of the rows above drop down to fill the

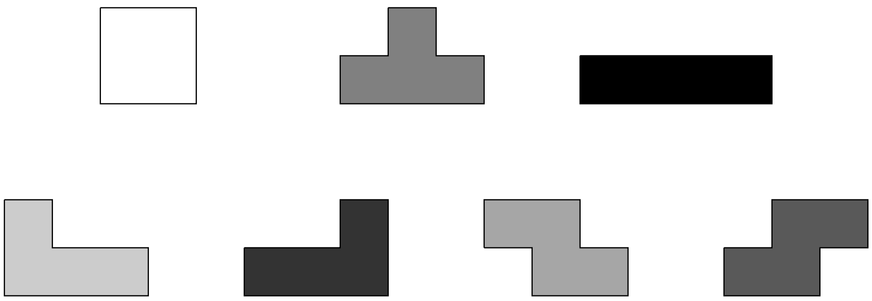

Figure 1: The Tetrominoes

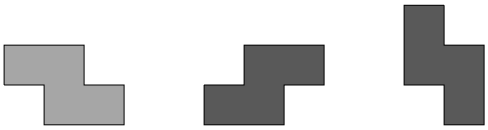

Figure 2: Z-Tetrominoes

gap. Figure 1 shows how this can open up gaps which had been buried, giving the player a second chance to fill these spaces. If the player does not fill the rows fast enough, eventually there will be no room on the board to place tetrominoes and the game will end.

Given just these instructions and a string of randomly selected tetrominoes falling very slowly, one can pack the tetrominoes densely enough to keep the game going for a long time. Tetris is interesting because players cannot correct their mistakes and because the rate at which pieces fall increases as the game progresses, forcing the player to make more mistakes. Would a perfect player be able to play Tetris indefinitely? Could one program a computer to play perfectly?

In his master's thesis [1], titled "Can You Win at Tetris?", John Brzustowski attempts to answer this question. Defining a "winning strategy" to be one that allows the player to continue a Tetris game indefinitely, he describes winning strategies for some very simple Tetris games, in which the same one, two, or three tetrominoes are presented to the player over and over. He also describes an algorithm the computer could follow to bring any Tetris game to an end.

Brzustowski proved that there is no winning strategy for Tetris **if the computer is is aware of and reacting to your moves**. This paper proves the slightly stronger result that the computer can win without being aware of the player's moves. This is done by producing a sequence of tetrominoes that will fill up the board no matter how the player places them.

A Very Special Tetris Game

We identify a Tetris game by the infinite string of pieces that appear on the screen during the game. Our first task is to prove that there is at least one Tetris game that can't be beaten. Since the odds of one particular Tetris game occurring in our lifetime are near zero, this may not seem like a useful result. However, we can extend this result to show that almost all Tetris games must eventually end.

Definition. *A* left handed *Z-tetromino is the Z-shaped tetromino shown on the left in Figure 2. The mirror image of a left-handed Z-tetromino (center) is called a* right handed *Z-tetromino. A* horizontal *Z-tetromino is one that is oriented so that its cells lie in three columns of the board, while the cells of a* vertical *Z-tetromino (right) lie in only two columns of the board. (Note that a Tetris player can easily convert horizontal tetrominoes to vertical ones, but can* *never turn left handed Z's into right handed ones.)*

We will show that a Tetris player presented with a sequence of Z-tetrominoes that are alternately right and left handed must eventually leave unfillable gaps in the rows of cells on the board. Such gaps prevent the cells on the rows they occupy from being removed from the board; an accumulation of gaps will fill the board and end the game.

Theorem 1 *The Tetris game consisting of only Z-tetrominoes alternating orientation will always end before 70,000 tetrominoes have been played.*

First we show that certain placements of Z-tetrominoes can only occur a finite number of times without ending the game. For instance, horizontal Z's contribute two cells to the column containing their center and only one cell to each of the columns on either side. This causes a "bump" to form in the center column, which the player can never fully smooth out. Similarly, if we number the columns of the board from one to ten starting with the leftmost column, vertical Z-tetrominoes placed with their leftmost cells in an even numbered column will cause unsmoothable bumps[1]. We can compute how many of these bumps can form before inequities in column height bring about the end of the game.

Lemma *In a Tetris game in which only Z-tetrominoes are presented to the player, no more than 120 Z-tetrominoes can be placed either vertically with their leftmost cells in an even-numbered column or horizontally in any column without losing.*

Proof: Number the columns of the Tetris board from one to ten. Let b_i be the total number of cells placed in the i^{th} column, let h_i be the number of horizontal Z-tetrominoes that contribute cells to the $(i-1)^{st}, i^{th}$, and $(i+1)^{st}$ columns, and let v_i be the number of vertical Z-tetrominoes dropped in the i^{th} and $(i+1)^{st}$ columns. Since all the tetrominoes in our game are Z's,

$$b_i = 2v_{i-1} + 2v_i + h_{i-1} + 2h_i + h_{i+1}.$$

If c_i denotes the number of cells in column i of the board, deleting a row leaves the difference $c_i - c_j$ fixed for any i, j. Hence $c_i - c_j = b_i - b_j$. Since the difference in the heights of the highest cells in columns i and j must be less than or equal to the height of the board, until the game is lost $|b_i - b_j| = |c_i - c_j| \leq 20$; we cannot add too many cells to column i without also adding some to j.

Since the tetrominoes must remain within the confines of the board, we know that $h_1 = h_{10} = 0$, so $b_1 = 2v_1 + h_2$; similarly, $b_{10} = 2v_9 + h_9$. Thus,

$$|b_2 - b_1| = 2v_2 + h_2 + h_3 \leq 20. \qquad (*)$$

Similarly, $2v_8 + h_8 + h_9 \leq 20$.

In general,

$$b_{i+1} - b_i = 2v_{i+1} - 2v_{i-1} + h_{i+2} + h_{i+1} - h_i - h_{i-1} \leq 20,$$

[1] These statements are equivalent to Brzustowski's Lemma 2 [1].

so $2v_{i+1} + h_{i+1} + h_{i+2} \leq 20 + 2v_{i-1} + h_i + h_{i-1}$. Letting $i = 3$ and applying equation $(*)$, we get:

$$2v_4 + h_4 + h_5 \leq 40.$$

Similarly, $2v_6 + h_6 + h_7 \leq 40$.

We conclude that $\sum_{i=1}^{10} h_i + \sum_{j=1}^{5} v_{2j} \leq 120.\Box$

From the lemma we see that in the Tetris game under consideration all players eventually reach a point beyond which they must place all the tetrominoes vertically in five "lanes", each lane two columns wide with odd numbered leftmost column (Figure 3). We now use the fact that the string of tetrominoes we are considering alternates orientation to prove that even placing the Z's in lanes results in the creation of undeletable rows and so brings an end to the game.

Suppose the topmost tetrominoes on the board are arranged in five non-empty lanes, each lane two columns wide. Consider the orientation of the top Z-tetromino in each lane. Since there are an odd number of lanes, there must be a majority of either right handed or left handed Z's on top.

Without loss of generality, we may assume that the majority of topmost Z's are right handed. Then, until a left handed Z is placed in a right handed lane, the total height of the left handed lanes will grow faster than the total height of the right handed lanes. Eventually (after at most 240 Z's are dropped) our player will be forced to place a left handed Z in a right handed lane or lose the game. When this occurs, a hole two cells deep and one cell wide forms in the rightmost row of the lane (See Figure 3).

Unless the player places a tetromino vertically between lanes or horizontally, this hole can never be filled; the two rows of cells containing the hole can never be removed from the board. When there are twenty rows on the board whose contents can't be deleted, the game will end.

Since a similar argument holds when a majority of the topmost Z's are left handed, the player must form such holes repeatedly. It is impossible to have more than 50 (ten per lane) holes like this on the board without ending the game.

We know from the lemma that a player can make at most 120 "exceptional" tetromino placements. Since each exceptional placement can fill at most two holes, the game might continue until $120 \cdot 2 + 50 = 290$ holes are formed. The player cannot place more than 240 Z's without forming a hole, so the game must end after at most $290 \cdot 240 = 69,600$ pieces are played.

This concludes the proof of Theorem 1: in a Tetris game in which only Z-tetrominoes of alternating orientation appear, any player will lose after placing at most 69,600 tetrominoes.

Other Tetris Games

We now have a loose upper bound on the amount of time a player can play one specific Tetris game. It would be nice to show that all Tetris games are necessarily finite, but this is not possible. A game in which only square

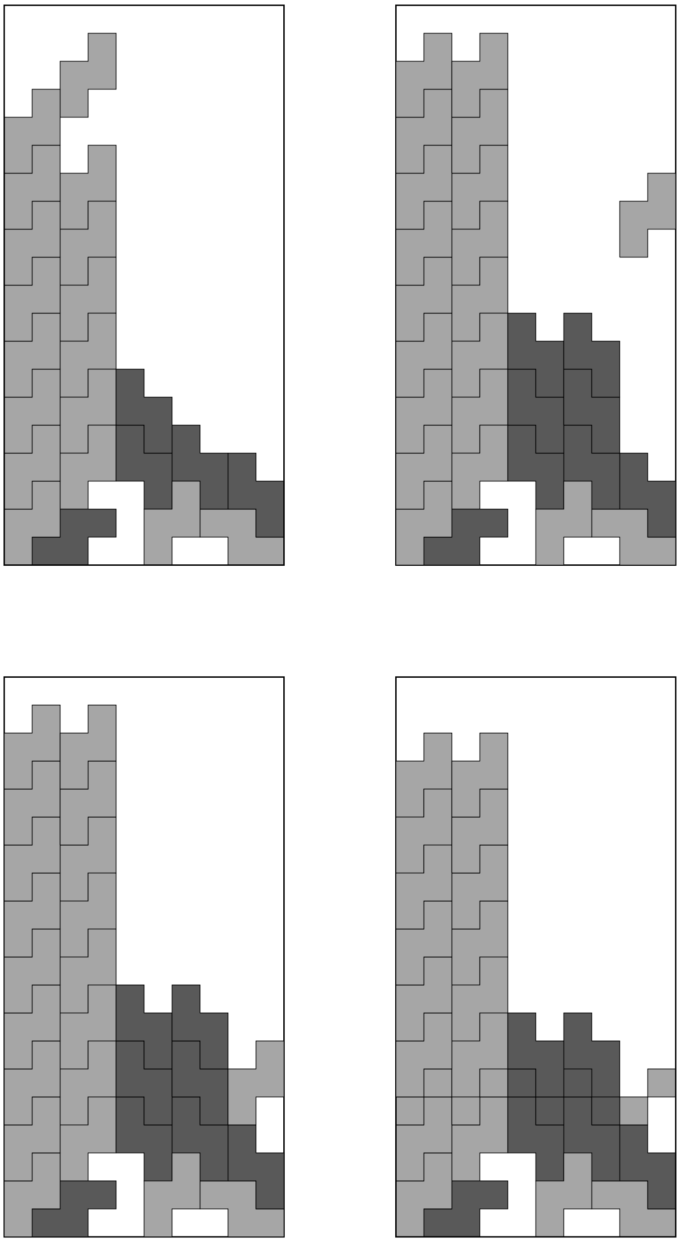

Figure 3: Stacking Z-Tetrominoes

tetrominoes (four cells arranged in a square) are dropped can go on forever. However, our methods do apply to *most* Tetris games.

Theorem 2 *Almost all Tetris games must eventually end.*

Proof: A randomly generated infinite string of tetrominoes will, with probability one, contain any given finite sequence of tetrominoes. Hence, we need only exhibit a finite string of tetrominoes that will force a loss to prove that almost all Tetris games will end. This string of tetrominoes will, of course, consist of alternating left- and right-handed Z-tetrominoes.

The methods of the previous proof still apply here, but we no longer know the initial condition of the board. Suppose our string of Z-tetrominoes occurs when there are a_i cells in column i; let b_i equal the number of cells added to column i beyond that point in the game, and let $d_i = b_i + a_i$ be the total number of cells placed in column i. For any i and j,

$$d_i - d_j = a_i - a_j + b_i - b_j \leq 20.$$

Thus $b_i - b_j \leq 20 + a_j - a_i$, and since $a_j - a_i \leq 20$, we get $b_i - b_j \leq 40$ for any i and j.

The above inequalities are very similar to those encountered in the proof of the lemma. In fact, if we go through the lemma's proof replacing $|b_{i+1} - b_i| \leq 20$ with $|b_{i+1} - b_i| \leq 40$, we can conclude that in a Tetris game with arbitrary initial conditions the bound on the number of exceptional placements is doubled (from 120 to 240). Using the techniques developed in the proof of Theorem 1, we conclude: whenever a string of $(240 \cdot 2 + 50) \cdot 240 = 127,200$ Z-tetrominoes of alternating orientations occurs in a Tetris game, the game must end.$\square$

We see that it is easy to lose a Tetris game, although it may take a long time.

Unanswered Questions

The bounds found in this paper are very loose; the number of alternating Z-tetrominoes needed to bring a Tetris game to a close may be closer to 1,000 than to 100,000. A more careful estimate of how many tetrominoes can be played between the formation of "holes" in lanes, or of the number of holes filled by each exceptional placement, would almost certainly reduce the bounds presented here.

Information on variants of the game Tetris and on Tetris games using the other four tetrominoes can be found in [1], but many questions remain unanswered. What sequences of tetrominoes will force a player to lose? What is the probability that one of these sequences will occur in the first n pieces dropped in a Tetris game? Is there an optimal strategy for Tetris? What if pentominoes are used in place of tetrominoes? I suspect that some of these problems will have simple solutions, while others may never be solved.

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