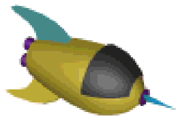
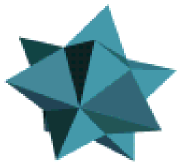


# The Shape of Space

***Curriculum Materials***



During the 1995 Summer Institute at The Geometry Center, a team of undergraduates, middle and high school mathematics teachers and Center staff developed curriculum materials supporting The Shape of Space video. The team developed a comprehensive curriculum, covering many areas related to the video. The following materials were selected for distribution based on their direct relevence to understanding the video itself.

For your convenience, a printable PostScript (14 Mb) version of all of these materials is available.

## Selected Materials



## Other materials available on the Web

Torus and Klein Bottle Games
: Play some of your favorite traditional games on the surface of a torus or a Klein bottle. Jeff Weeks created interactive Java games (Tic-Tac-Toe, Chess, Jigsaw puzzles, Mazes, Word Searchs and Crossword puzzles) for learning about these surfaces.

Topology Materials
: Chaim Goodman Strauss' ...

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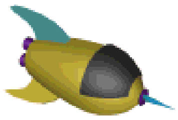
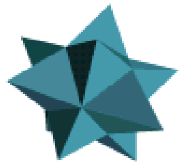


# The Shape of Space

***Curriculum Materials***



# Contributing Developers

### Undergraduate students:

- Barbara Foster
- Jodi Schneider
- Daniel Verinder

### Teachers:

- Jim Huber
- Laura Moss
- Jerry Ter Eick

### Geometry Center staff:

- Chaim Goodman-Strauss
- Lori Thomson
- Stuart Levy

**Up:** *The Shape of Space Curriculum Materials*



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Created: Tuesday, 01-Apr-97 17:45:20 --- Last modified: Thursday, 08-May-97 13:50:15 CDT

# Dimensions - Exploring Flatland

### The Shape of Space Curriculum Materials

## Objectives

- Students will learn several ways in which living in two dimensions is more restrictive than living in three dimensions.

## Materials

- *Flatland* Activity Sheet
- Hints and Answers to Flatland questions.
- String, paperclips, and otherthings flatlanders can use or be made of.
- The Shape of Space video

## Activities

Have students explore what it would mean to be a two-dimensional being instead of a three-dimensional being. Ask a volunteer to come up to the board and draw a two-dimensional animal (a flatlander). The most common mistake that people make is with the placement of the flatlander's eyes. A person's first attempt at drawing a flatlander usually has the eyes on the inside of the head:

If this happens, ask the students, "How will the flatlander see another flatlander?" Also, people usually draw the mouth on the inside of the head, too. Then how will the flatlander eat? A reasonable second attempt at a flatlander head could look something like this:

Either continue discussion as a class, or let students explore possibilities in small groups using the questions on the activity sheet provided.

## Wrap-up

At the end of the activity, look again at the flatlanders in The Shape of Space video. Ask the students some questions:

- How did the makers of the video deal with the flatlander issues that the students explored today?
- Where did they place the eyes?

- How would it look to us 3-D people if the flatlander took a swim in the lake? Could we still see him when he went under water?
- Would the flatlander get wet?
- How many other things do you notice about how the flatlanders were designed?

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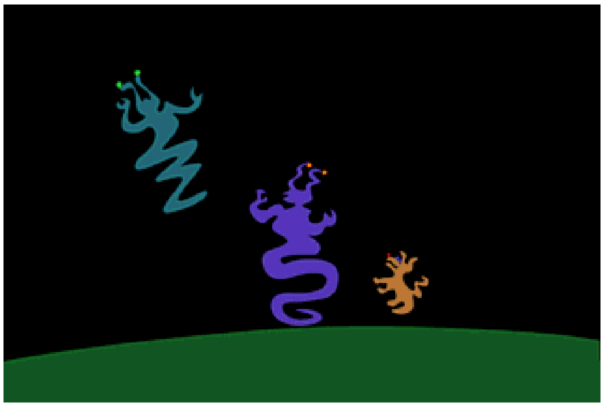
# Dimensions - Exploring Flatland

### Activity Sheet

*The Shape of Space Curriculum Materials: Dimensions*


## What *would* life in Flatland be like?



The Shape of Space has one interpretation of what Flatlanders would look like. But this is just one possible drawing of some Flatlanders. Flatlanders have found their way into lots of books, like *An Episode of Flatland*, *Sphereland*, *Planiverse*, and *Flatland*. In *Flatland*, the main character is a square who gets sent to prison after his experiences with three dimensions. He has difficulty imagining three dimensions until an encounter with Lineland, a one-dimensional world whose inhabitants are lines.

## Be Creative!

In groups, draw yourselves as Flatlanders and write a story about life in Flatland. Here are some questions to get you thinking:

Can a flatlander tie a knot? *[ a hint | an answer ]*

How would Flatlanders carry things? *[ a hint | an answer ]*

What would a fence look like in Flatland? What about a picket fence? (The picket fence should keep out other Flatlanders but allow them to see the flowers inside.) *[ a hint | an answer ]*

How would Flatlanders play baseball? volleyball? tennis? What would be the difference between these three games? *[ a hint | an answer ]*

How would a Flatlander eat/digest food? *[ a hint | an answer ]*

Make your own list of questions to explore.

# Dimensions - Exploring Flatland

**hints**

## Question:

Can a flatlander tie a knot?

## Hint:

Place a piece of string between two pieces of paper. (Clear paper or transparencies work well.) Your challenge is to pass one string over the other while keeping the string in the "plane" between the two pieces of paper. Pressing the pieces of paper tightly together may help you to imagine the restrictions of the plane.

---

## Question:

How would Flatlanders carry things?

## Hint:

**For a bag:** How would you make a bag two dimensional? Try trapping some paper cut-outs in the plane a desk or table. In Flatland these pieces could not fall off the table. They could only move in left-right and north-south directions on the table.

**For a handle:** Think about how toy trains connect. How would this translate into two dimensions?

---

## Question:

What would a fence look like in Flatland? What about a picket fence? (The picket fence should keep out Flatlanders but allow them to see the flowers inside.)

## Hint:

If you have an idea of how to make a 2-D bag, then think about how a fence is like a bag. Should your fence be closed or open? Decide and explain your decision.

If you don’t know how to make a bag, then try to trap paper cut-outs on a table. Remember that in Flatland, your cut-outs would not fall off the table--they would always stay in the plane.

To make a picket fence or bag you can see through, think about what sort of 3-D objects you can see through. A mesh bag, for example, has cloth "missing" from the bag. How is it held together? With pieces of string connected to one another. A picket fence in Flatland would really be made out of lots of individual pickets. These pickets--unlike most fences you’ve probably seen--wouldn’t need to be connected to one another.

---

## Question:

How would Flatlanders play baseball? volleyball? tennis? What would be the difference between these three games?

## Hint:

Draw a 3-D game from the side. Flatlanders would play with a 2-D object, just like the Flatlander’s sun in the Shape of Space video.

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## Question:

How would a Flatlander eat/digest food?

## Hint:

Flatlanders’ bodies would fall into two parts if they were cut by a digestive tract. Two dimensional fasteners which are analogous to magnets, zippers, buttons, or velcro might help your Flatlanders stay whole. The fastening techniques used for handles might also be useful. Try modeling a Flatlander’s digestive tract out of paper.

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# Dimensions - Exploring Flatland

### answers

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### Question:

Can a flatlander tie a knot?

### Answer:

No. Try making a knot with a piece of string without passing one over the other.You can't do it. (See also Planiverse bio section--good diagrams)

A souped-up version of this demonstration could be done with a piece of rope pressed between two pieces of plexiglass (or another clear surface. Leave the bitter ends of the rope free of the plexiglass so that you can move them. You can now move the rope in the left-right direction and topwards-bottomwardsdirection, but not out of the plane. Thus this model is a demonstration of a 3-Drope confined to two dimensions.

---

### Question:

How would Flatlanders carry things?

### Answer:

**Bag:** A line--whether straight or curved--forms a "wall" in 2D. Thus an open curve would function as a bag in 2D. Flatlanders could put their belongings into the bag through the open end but would need to make sure that nothing fell out of the bag through that opening.

**Handle:** Two lines can interlock in a C shape and its mirror image. These or similar interlocking shapes would make good handles.

---

### Question:

What would a fence look like in Flatland? What about a picket fence? (The picket fence should keep out Flatlanders but allow them to see the flowers inside.)

### Answer:

Like a bag, a fence would be formed of a line or several lines, whether curved or straight. A picket fence could be made out of many points with spaces between them or by line segments separated by space. This fence would keep out anything larger than the distance between the points but would let Flatlanders see their garden. Flatlanders would probably prefer line segments to points: points would be very sharp and might cut Flatlanders in two.

---

### Question:

How would Flatlanders play baseball? volleyball? tennis? What would be the difference between these three games?

### Answer:

Each game would be played with some sort of 2-D disk. The object of the game would be to get the disk

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### Question:

How would a Flatlander eat/digest food?

### Answer:

A digestive tract made out of two parallel straight lines would be deadly for a Flatlander. The digestive tract would divide the Flatlander's body into three parts: the digestive tract itself, the body to left of the digestive tract, the body to the right of the digestive tract. These pieces either need to be held together with magnets or some similar objects, or the digestive tract must be made out of lines which hold themselves together. Instead of straight lines, a combination of U, C, or similar shapes would hold the Flatlanders together.

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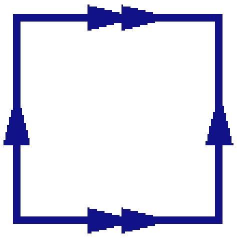
**Up:** *The Shape of Space: Dimensions - Exploring Flatland*

*The Geometry Center Home Page*

# Surfaces - Gluing Diagrams And Flatlander Games

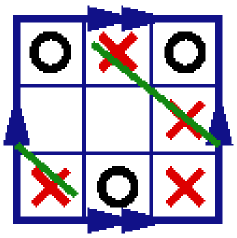
**The Shape of Space Curriculum Materials**

## Gluing Diagrams



The purpose of the gluing diagram activity is for students to make the connection between a gluing diagram and the surface of a three dimensional object. By physically cutting out gluing diagrams and taping them together, students get a feel for properties of a disc, a cylinder, a Moebius strip, a torus, a Klein bottle, and the Projective Plane.

## Flatlander Games



If students understand the notation used in gluing diagrams, they can play Tic-Tac-Toe on many surfaces. Tic-Tac-Toe is a game that many students have mastered at an early age. Two good players will always end up at a draw. But by getting rid of the boundaries, we change the game and make it interesting again!

## Additional templates for activities

A. **Disc** *(square fundamental domain)*
B. **Cylinder** *(square fundamental domain)*
C. **Moebius strip** *(square fundamental domain)*
D. **Torus** *(square fundamental domain)*
E. **Klein bottle** *(square fundamental domain)*
F. **Projective Plane** *(square fundamental domain)*
G. **Disc** *(hexagonal fundamental domain)*
H. **Torus** *(hexagonal fundamental domain)*
I. **Klein bottle** *(hexagonal fundamental domain)*

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Created: Tuesday, 01-Apr97 17:45:20 --- Last modified: Thursday, 15-May-97 17:39:00 CDT

# Surfaces - Gluing Diagrams

### The Shape of Space Curriculum Materials

## Objectives

- Students will be able to match gluing diagrams with a picture and the name of the surfaces they represent for the disc, annulus, Möbius band, torus, Klein bottle and projective plane.

## Materials

- *Gluing Diagrams* template **G1** or templates **G2 and G3** or templates **G2b and G3b**.
- *Gluing Diagrams* Activity Sheet
- Scissors and tape or glue (tape works better).

## Activities

Introduce the idea of topology to students, for example:

> In geometry, you learned that a square and a triangle are different shapes. The number of corners an object has makes a difference when you talk about its geometry. In *topology* though, number of corners doesn't matter. In fact, you can stretch, bend and deform a shape as much as you want to without changing its topology. The only things we can't do to it, are to puncture any holes in the shape or to fill in any holes.

Hand out the templates and the activity sheet to the students. The first page contains all six of the gluing diagrams used in this lesson, but some of them are difficult to work with on paper because paper doesn't stretch. For example, it is difficult to make a Möbius band out of a square piece of paper. To make the diagrams easier to work with, two additional pages are provided with the diagrams stretched somewhat already.

Explain the concept of a gluing diagram, for example:

> In a Gluing diagram, arrows or other markings are used show where a surface should be connected up with itself. A square without markings is just a square. [Draw a square.] It has boundaries in all directions. Now if we connect the left side to the right side [Mark left and right sides with arrows going same direction], then a flatlander living in this space could go out the right side and come back in on the left. In fact, the flatlander could travel for ever in that direction without coming to a boundary. The top and bottom are still boundaries, though, so the flatlander couldn't travel far in either of those directions. What if the arrows are drawn in opposite directions? [Change the direction of one of the arrows.] What might

this mean? [Let the students speculate.] Now, when a flatlander goes out the right hand side, it comes back in on the left upside-down (mirror-reversed). Right-handed flatlanders become left-handed. [You may want to let students discuss what it would be like to be mirror-reversed. Would you feel like you're reversed, or would it seem like everyone else was reversed?]

Have the students cut out the gluing diagrams. Then have them fold or roll the paper so that the arrows line up properly. For best results, match the sides with one arrow first, then the sides with two arrows. Remind them that they are changing the geometry of the object, but not the topology.

Once the object is taped together, see if they can match it up to the picture on the activity sheet. They will probably need a lot of guidence from the teacher.

## Hints and Answers

1. **F. d. annulus**
   This is not difficult to glue on either of the templates. They may recognize the object as a tube or a cylinder. A surface with this topology is called an "annulus."

2. **D. a. disc**
   No gluing is required of this object. The square is topologically equivalent to a circle or a "disc."

3. **A. e. Möbius band**
   This object is easier to glue using the second page of templates (the long and narrow gluing diagram). The shape you get is a "Möbius band" or "Möbius strip." This shape may be familiar to some students. It has many interesting properties of its own. For instance, how many sides does it have? [one] How many edges? [one] What do you get if you cut a Möbius strip down the middle?

4. **B. f. Torus**
   This object is easier to glue using the second page of templates (the wide gluing diagram). Students may recognize the shape as a donut, an innertube or a "torus" (not to be confused with the Zodiac sign "Taurus" which is spelled differently). This topology is often used in video games (go off one side come in the other).

5. **C. b. Klein bottle**
   To connect up this figure, the surface must pass through itself. This is a legal operation in topology. However, it is impossible to do with paper. Students will have to use their imaginations for this one. You may want to let the students struggle with this one using the square gluing diagram. The second gluing diagram is shaped like a parallelogram and suggests where you might want to cut a hole to allow the surface to pass through itself. But cutting holes changes the topology! If you go this route, stress to the students that they have to imagine that the surface does not have a hole, but it does pass through itself. This surface is called a "Klein bottle." The Geometry Center has many nice pictures of the Klein bottle in their Topological Zoo. Like the Möbius band, the Klein bottle has only one side. So, a Klein bottle can't hold any water!

6. **E. c. Projective plane**
   In order to be glued up, this surface must pass through itself also. The easiest way to put this one together is to cut a diagonal line from any corner to the center of the square. Then twist the square

about the center so that the arrows line up properly. The two sides to the cut that you made should line up again. Imagine that they are connected together *through* the other part of the surface. This representation is similar to the crosscap image of the projective plane found in The Geometry Center's Topological Zoo.

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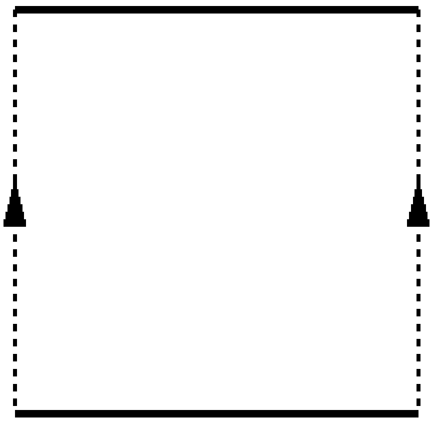
# Gluing diagrams (page 1)

*The Shape of Space Curriculum Materials: Surfaces*

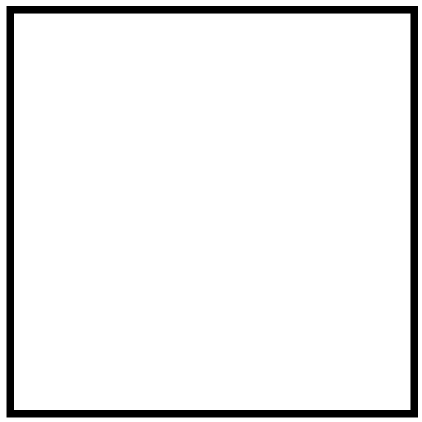


Cut these out and glue or tape them together so that the arrows line up. Can you name the shape that you get when they're connected? (Solid lines are boundaries; don't tape them to each other.)

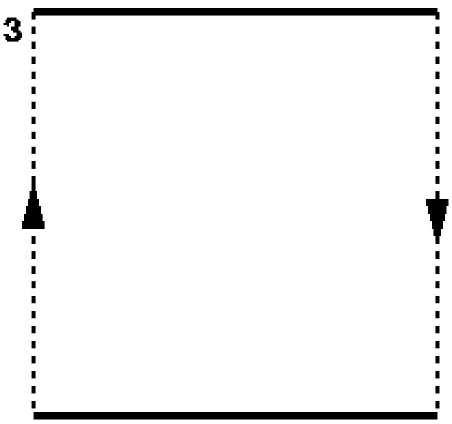
1



2



3



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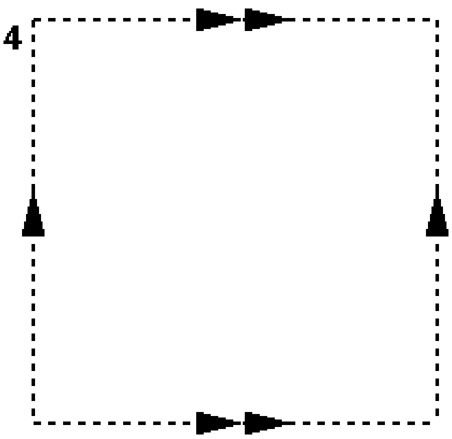
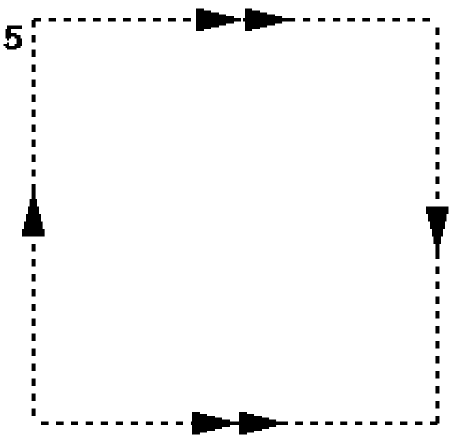
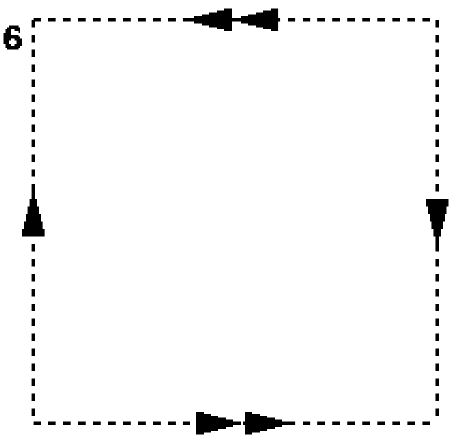


5



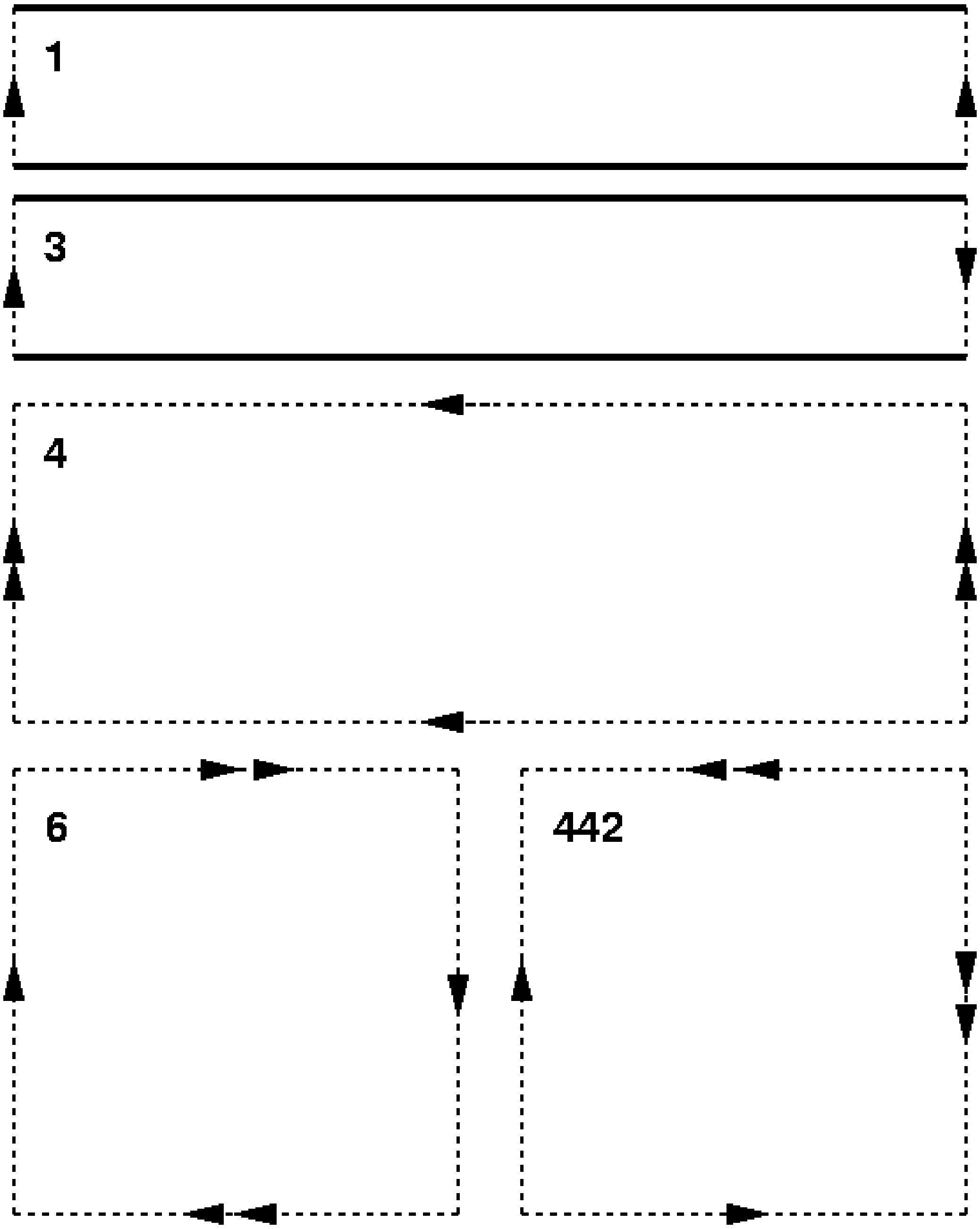
6



# Gluing diagrams (page 2)

*The Shape of Space Curriculum Materials: Surfaces*

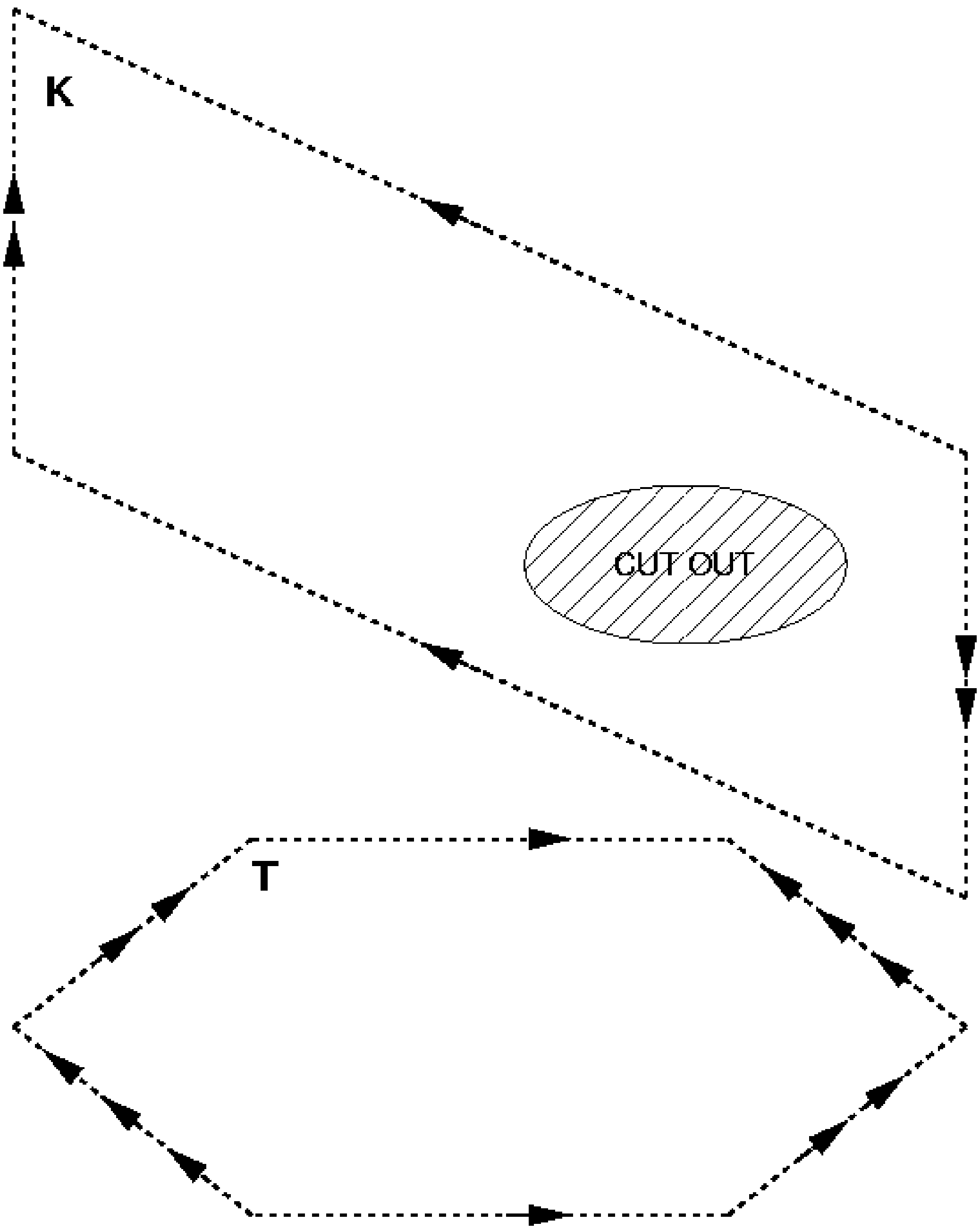

Cut these out and glue or tape them together so that the arrows line up. Can you name the shape that you get when they're connected? (Solid lines are boundaries; don't tape them to each other.)



# Gluing diagrams (page 3)

*The Shape of Space Curriculum Materials: Surfaces*


Cut these out and glue or tape them together so that the arrows line up. Can you name the shape that you get when they're connected? (Solid lines are boundaries; don't tape them to each other.)



# Gluing diagrams (page 2b)

*The Shape of Space Curriculum Materials: Surfaces*



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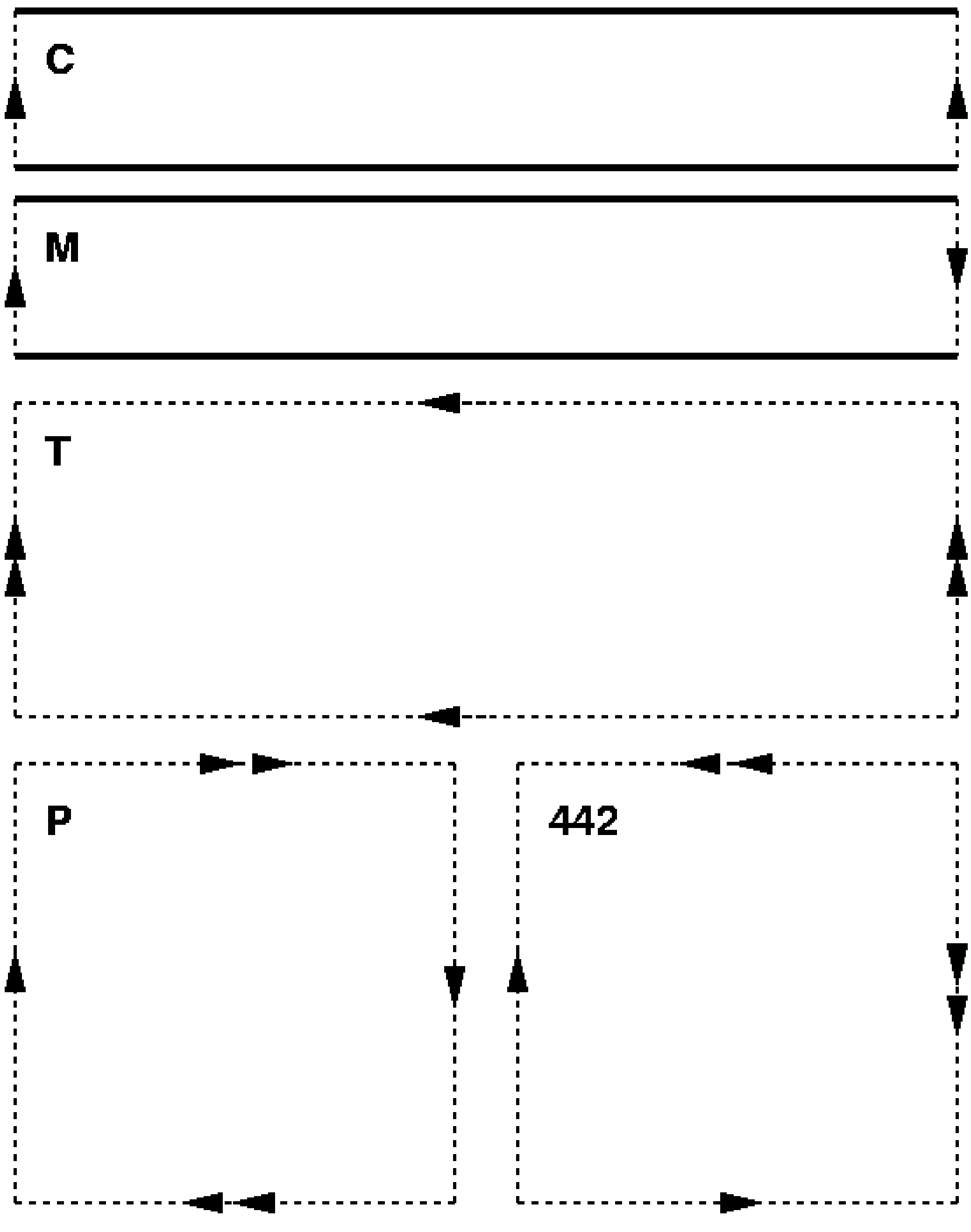
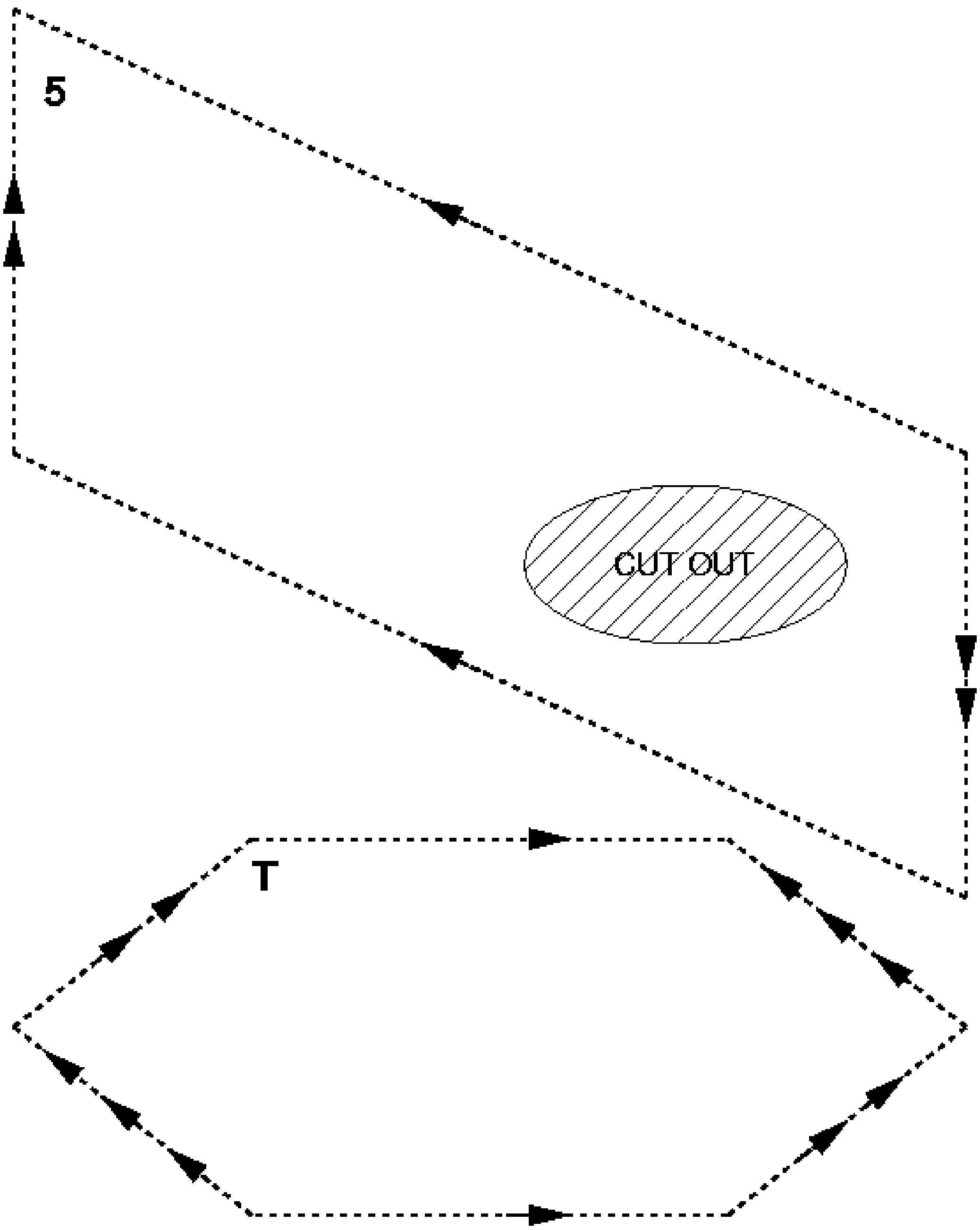
Cut these out and glue or tape them together so that the arrows line up. Can you name the shape that you get when they're connected? (Solid lines are boundaries; don't tape them to each other.)



# Gluing diagrams (page 3b)

*The Shape of Space Curriculum Materials: Surfaces*


Cut these out and glue or tape them together so that the arrows line up. Can you name the shape that you get when they're connected? (Solid lines are boundaries; don't tape them to each other.)



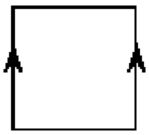
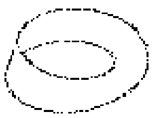
# Surfaces - Gluing Diagrams

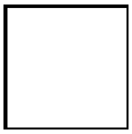
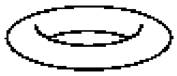
## Activity Sheet

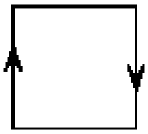
*The Shape of Space Curriculum Materials: Surfaces*

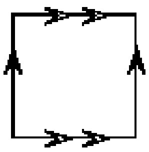
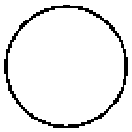


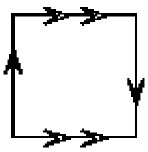
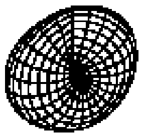
Match the gluing diagram with the surface it creates and the name of the surface.

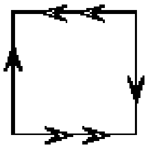
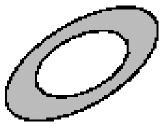
**1.**  A.  a. Disc

**2.**  B.  b. Klein bottle

**3.**  C.  c. Projective plane

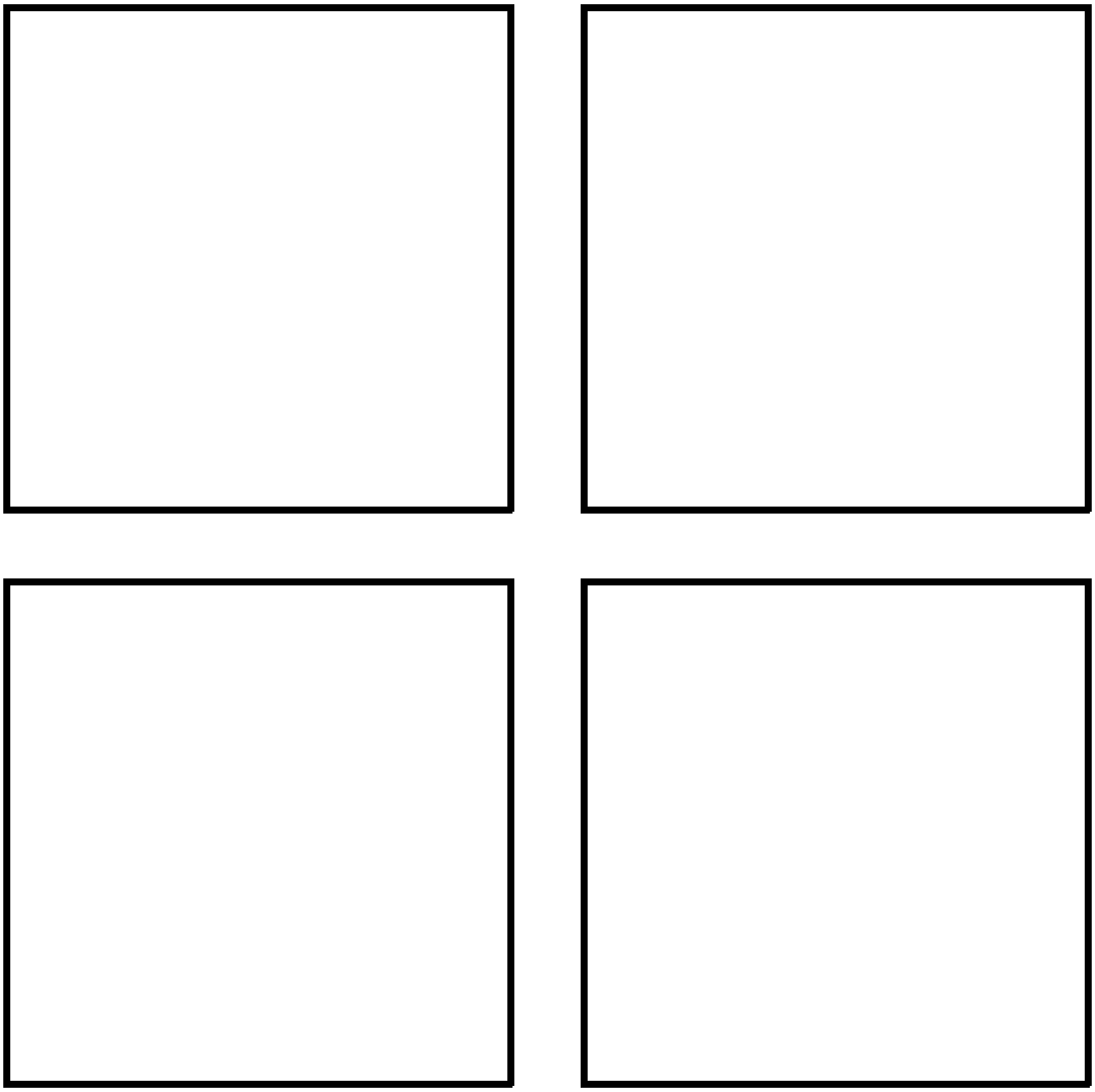
**4.**  D.  d. Annulus

**5.**  E.  e. Möbius band

**6.**  F.  f. Torus

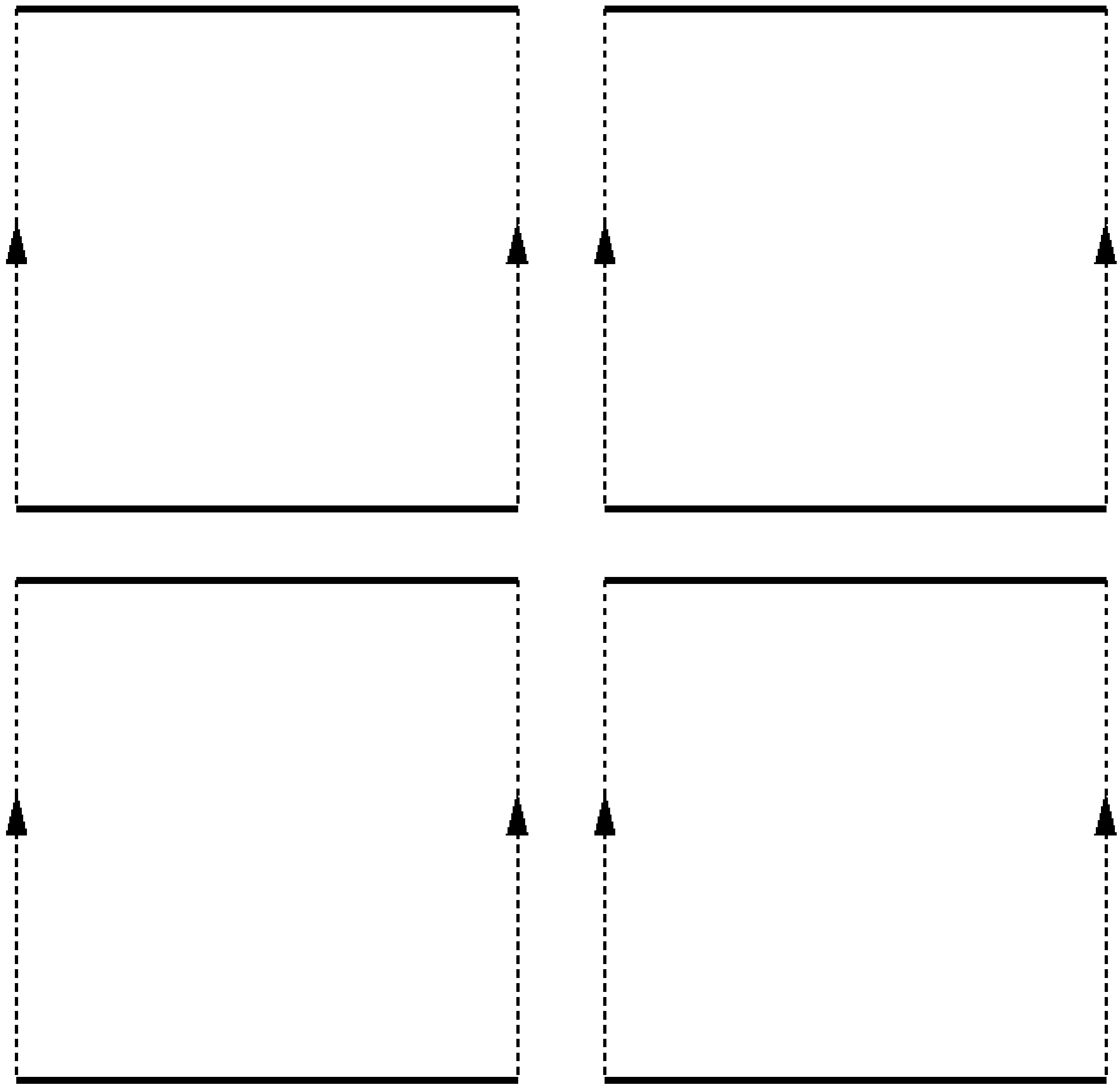
# Disc (square fundamental domain)

This surface has boundaries on all sides.



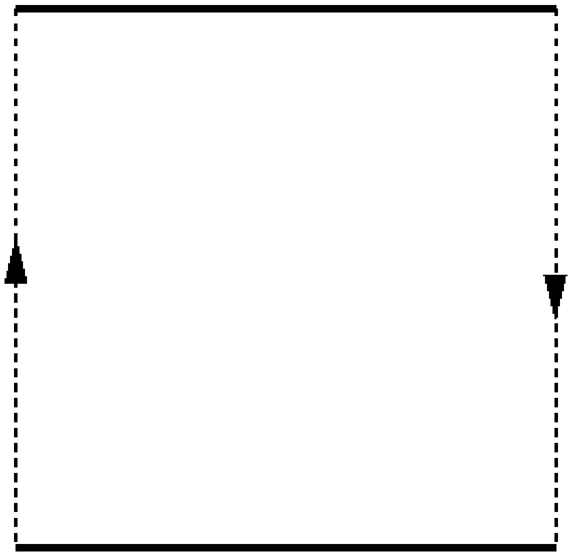
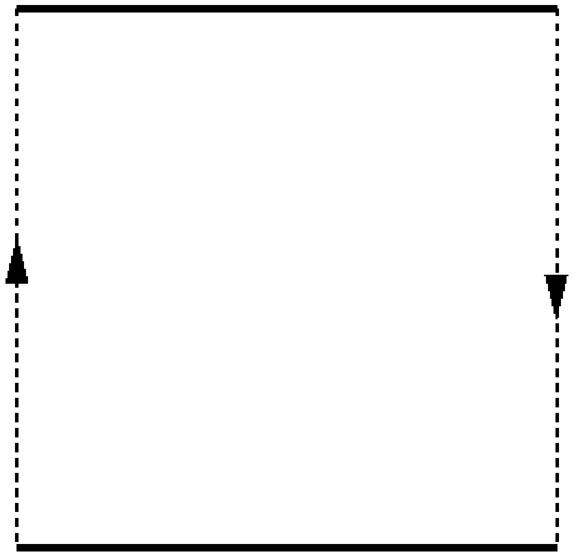
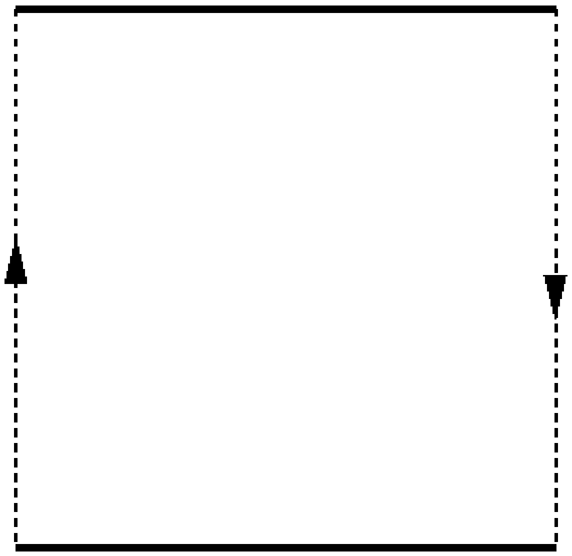
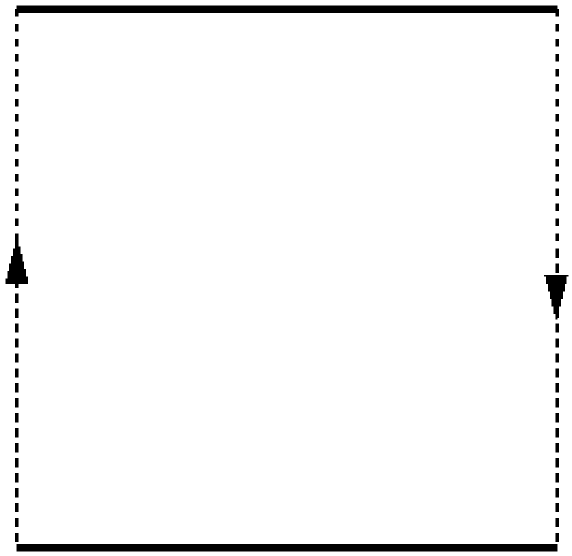
# Cylinder (square fundamental domain)

This surface has a boundary in one direction only. The other sides connect in the same direction (no flip).



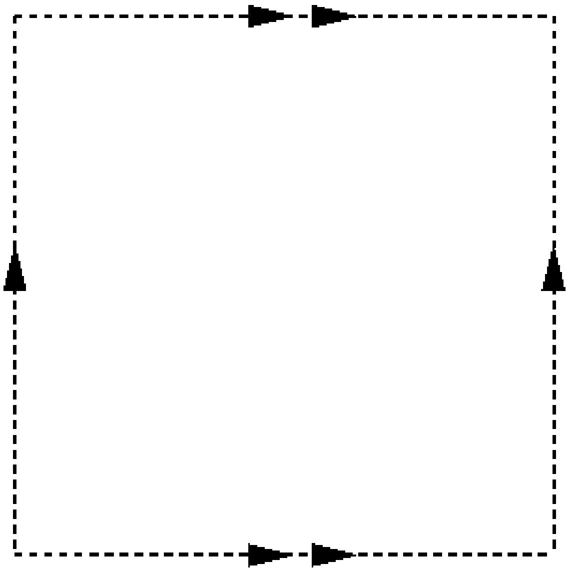
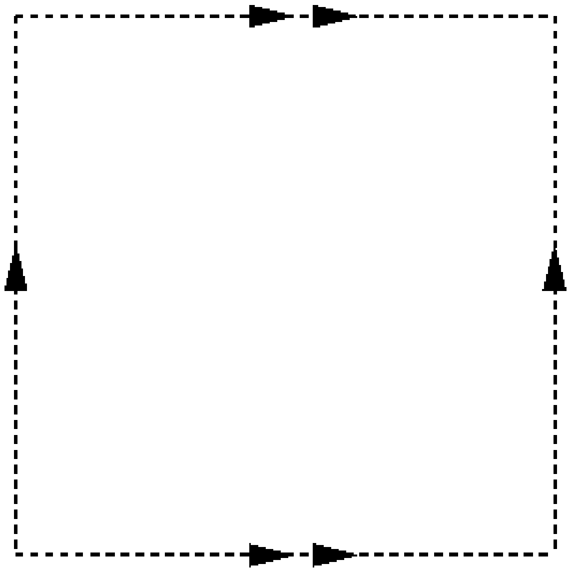
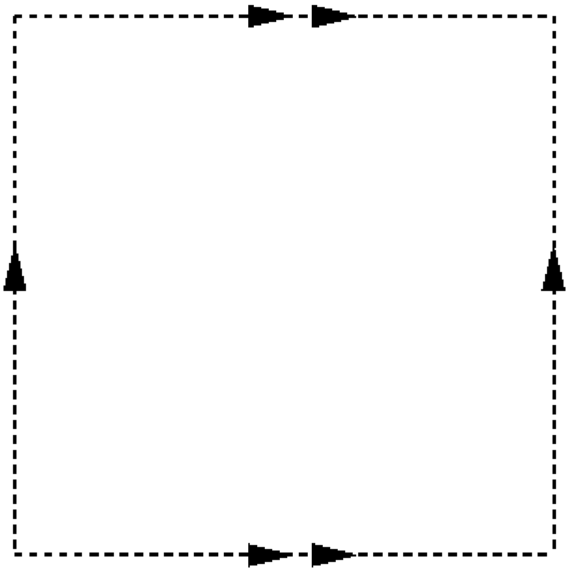
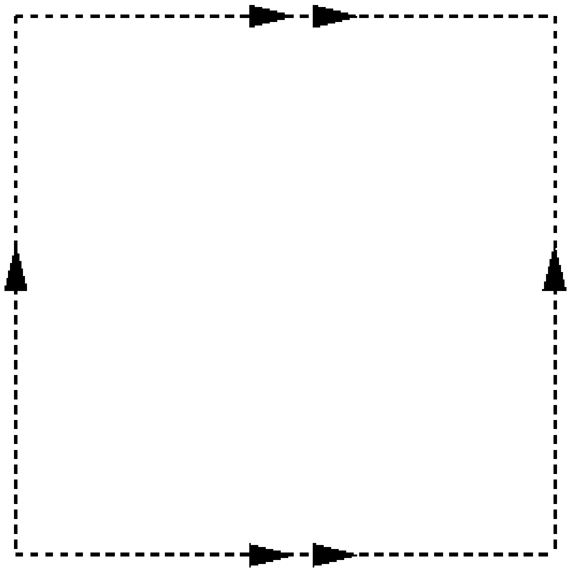
# Moebius strip (square fundamental domain)

This surface has a boundary in one direction only. The other sides connect with a flip.

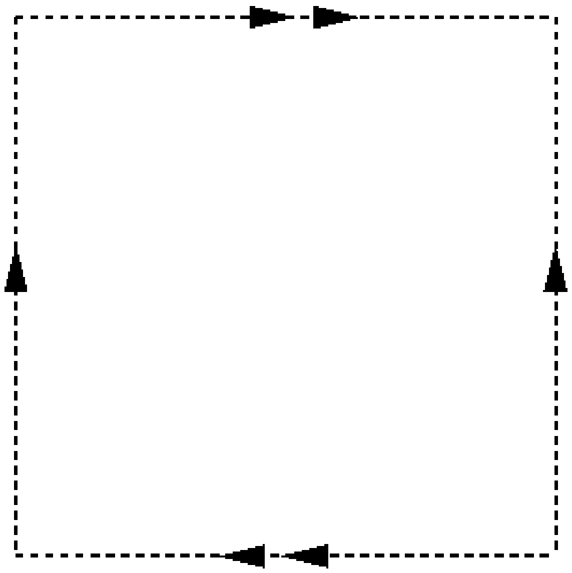
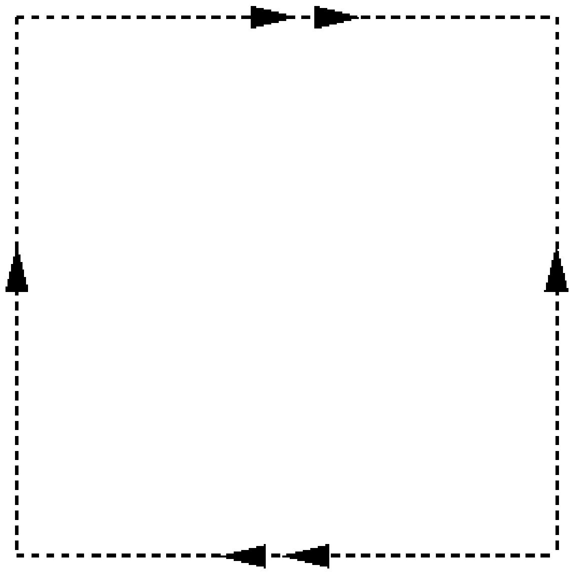
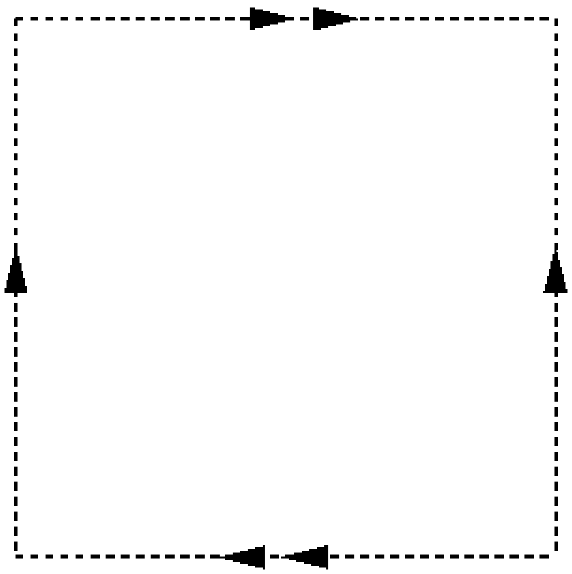
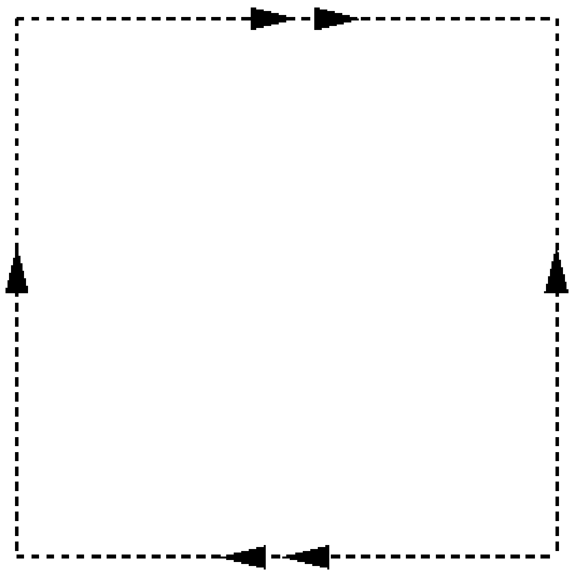
# Torus (square fundamental domain)

This surface has no boundaries. The left and right connect simply as do the top and bottom sides.

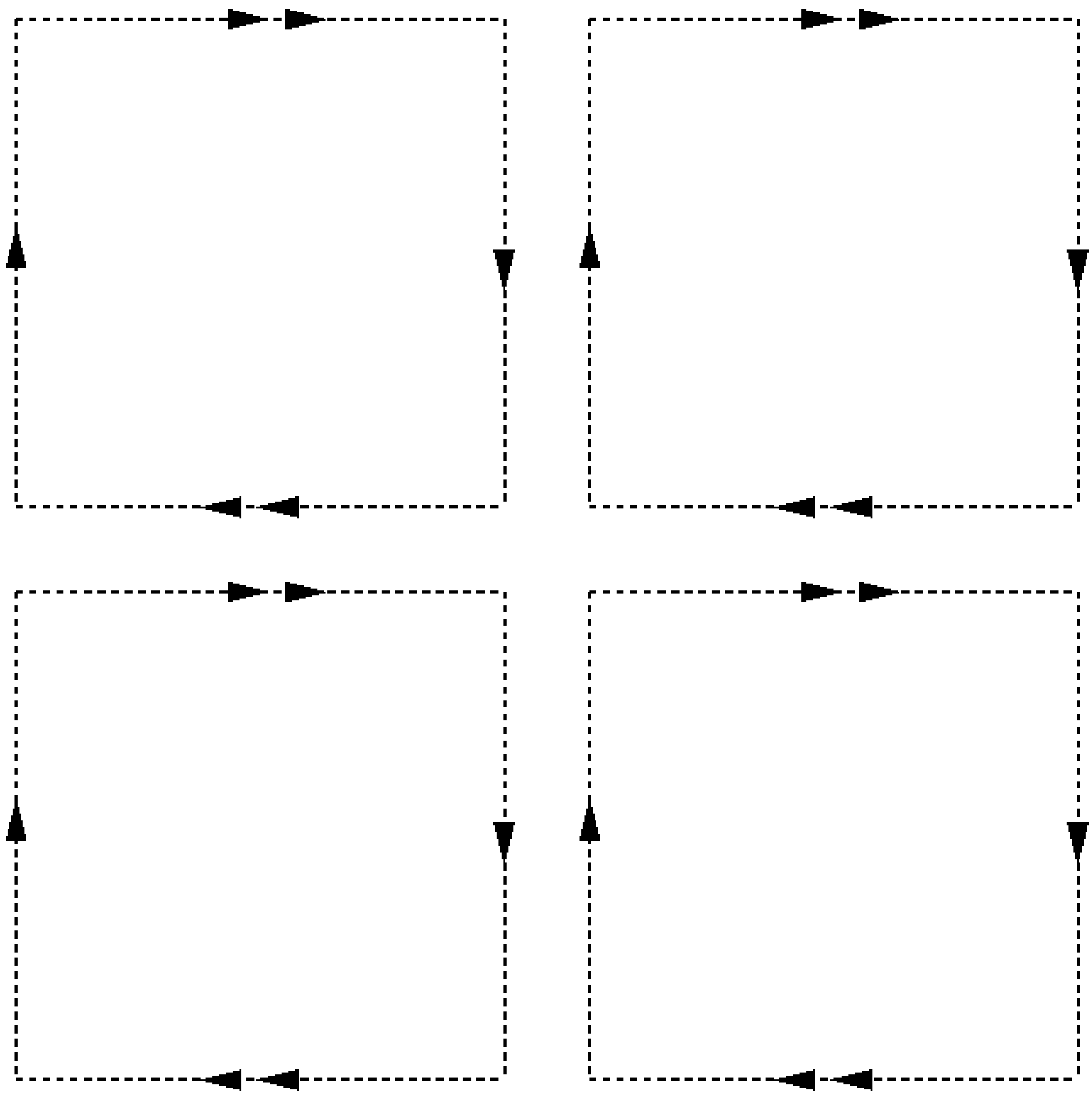
# Klein bottle (square fundamental domain)

This surface has no boundaries. One pair of sides connect simply like in the torus, but the other pair connect with a flip.

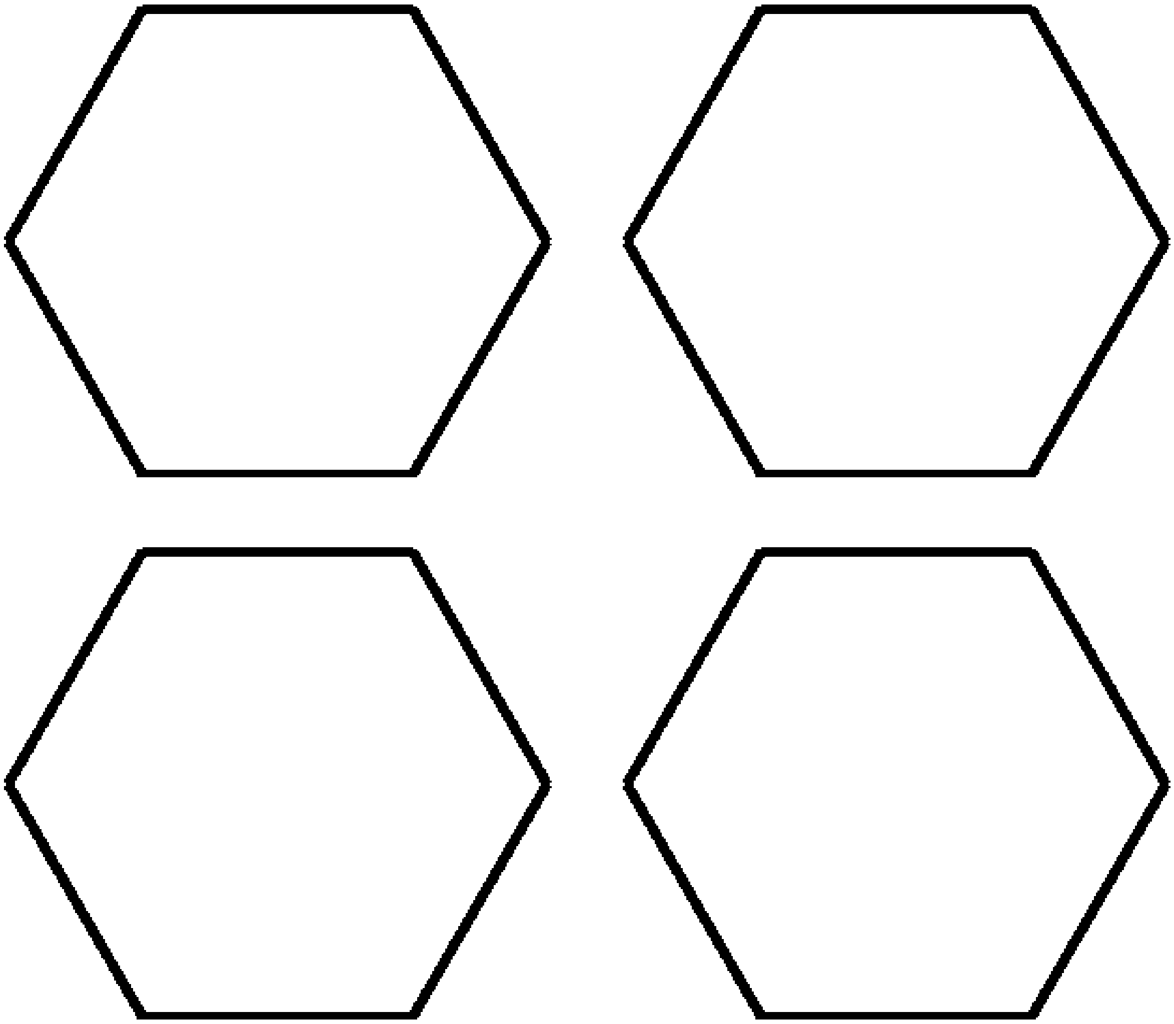
# Projective Plane (square fundamental domain)

This surface has no boundaries. The connections in both directions are flipped.



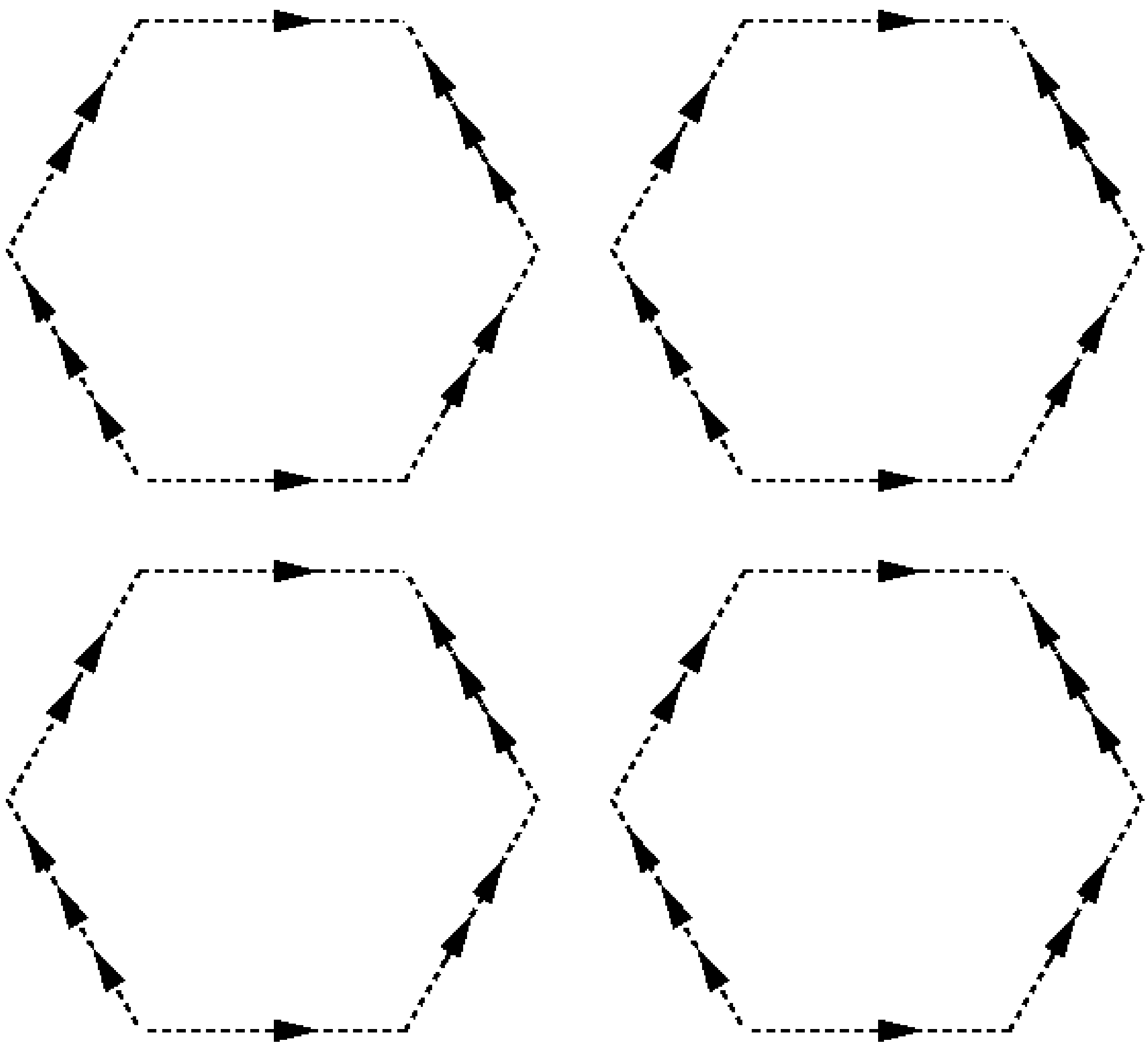
# Disc (hexagonal fundamental domain)

This surface has boundaries on all sides.



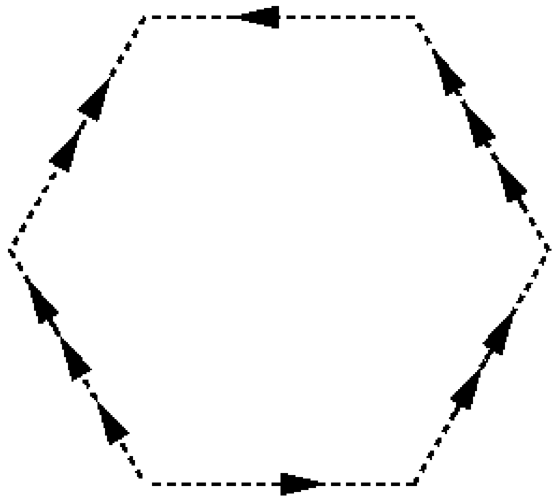
# Torus (hexagonal fundamental domain)

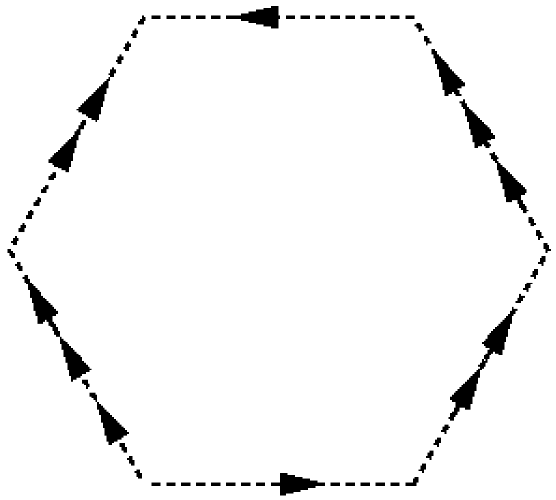
This surface has no boundaries. All three pairs of sides connect simply (no flip).

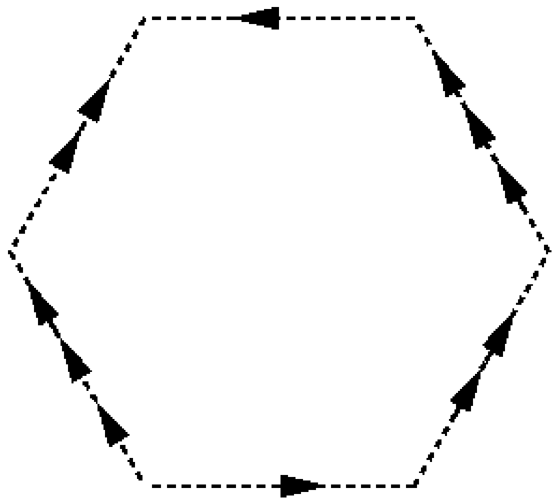


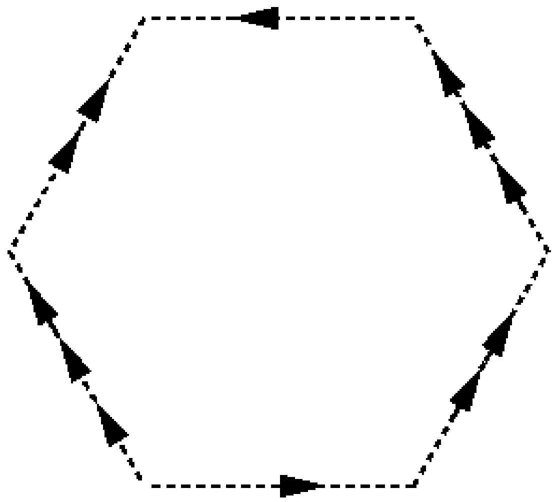
# Klein bottle (hexagonal fundamental domain)

This surface has no boundaries. Two pairs of sides connect simply, while one pair of sides connects with a filp.









# Surfaces - Flatlander Games

### The Shape of Space Curriculum Materials

## Objectives

- Students will discover how strategies of Tic-Tac-Toe change when played on a torus or a Klein bottle. The ability will be measured by answering several questions about the strategies.
- Immeasureable outcomes: Students will get a feel for how flatlanders might feel living in a torus or klein bottle.

## Materials

- *Torus Tic-Tac-Toe* Activity Sheet

## Activity

Play a sample game as a class (half the class against the other half) to remind the students of the notation used on gluing diagrams. Then let students play tic-tac-toe on their activity sheets in pairs. Consider the extensions to this activity listed below to further enrich the understanding of these surfaces.

## Extensions

If you have a Web browser that runs Java applications (e.g. Netscape 3.0 or later), you can play Tic-Tac-Toe and other games on a torus or a Klein bottle at http://www.northnet.org/weeks/TorusGames/TorusGames.html

Try playing games on other surfaces:

A. **Disc** *(square fundamental domain)*
B. **Cylinder** *(square fundamental domain)*
C. **Moebius strip** *(square fundamental domain)*
D. **Torus** *(square fundamental domain)*
E. **Klein bottle** *(square fundamental domain)*
F. **Projective Plane** *(square fundamental domain)*
G. **Disc** *(hexagonal fundamental domain)*
H. **Torus** *(hexagonal fundamental domain)*
I. **Klein bottle** *(hexagonal fundamental domain)*

You may wish to ask your students to think about whether their favorite board games, like chess, checkers or Othello, can be played on a torus or klein bottle.

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# Surfaces - Torus Tic-Tac-Toe

### Activity Sheet

*The Shape of Space Curriculum Materials: Surfaces*


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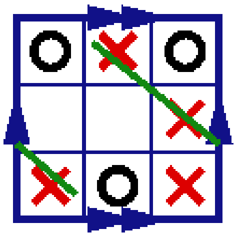
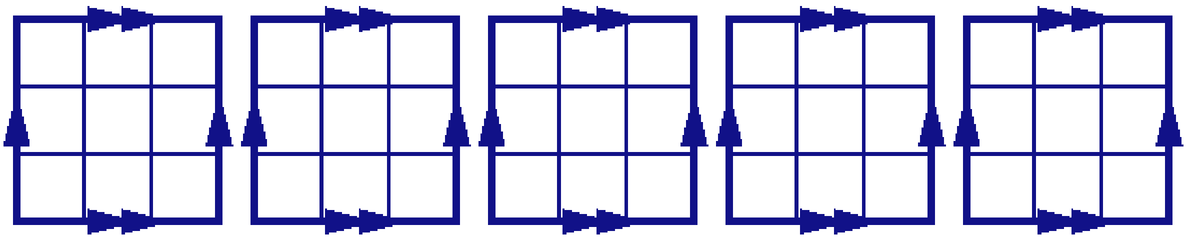
Think you've got Tic-Tac-Toe all figured out? ... Try ***Torus***-Tic-Tac-Toe!



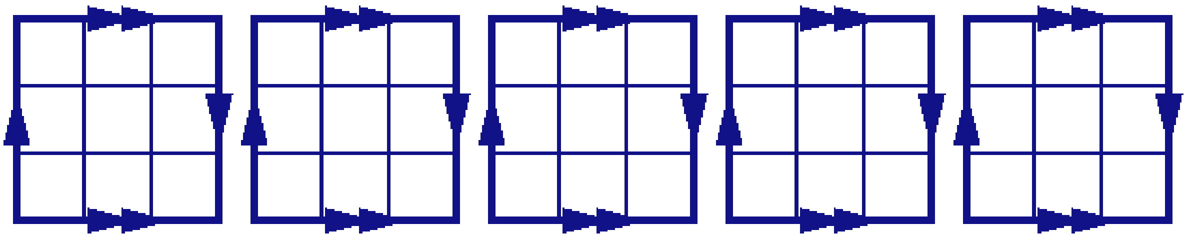
Glue each pair of opposite sides together, as shown to the left. Now there are more ways to get three in a row! (and more ways to lose -- keep your eyes open!) *X wins in this torus tic-tac-toe game.*

If you have a Web browser that runs Java applications (e.g. Netscape 3.0 or later), you can play Tic-Tac-Toe and other games on a torus or a Klein bottle at http://www.northnet.org/weeks/TorusGames/TorusGames.html

Play a few games of tic-tac-toe on a torus:



Now try it on a Klein bottle. This is a bit trickier because of the flip.



## Questions:

1. Does it matter which player goes first? Why or why not?
2. Is there a "best place" for the first marker? Where or why not?
3. For each possible first move, how many essentially different second moves are there?
4. Can you come up with a strategy so that you will always win?
5. Can there ever be a draw in this game?

# Geodesics - Line of Sight

**The Shape of Space Curriculum Materials**

## Objectives

- Students will be able to use tiled pictures to show a flatlander's line-of-site.

## Materials

- Template A
- Colored pencils (6 color minimum), scissors, tape, straight edge

## Activity 1

1. Cut out one square from Template A.
2. With colored pencils and straight edge, draw straight lines connecting A to each of the eight other points of the square. For best results, lines with different slope should be colored a different color; lines with the same slope should be colored the same color.
3. Label each line with its correct slope. (Given: the slope of AD = 0, and the slope of AB is undefined.)
4. Circle the points in red which are out of the line of sight of A.

## Activity 2

1. Cut out a second square from Template A.
2. Tape the second square to the left of the first square as shown:
3. Draw lines connecting A or the copy of A to each of the other points of either square. Lines with the same slope should be colored the same.
4. Use the following table to list the different slopes drawn on the figure.
5. In blue, circle all the points which are out of sight of A, or the copy of A.

| | | | | | |
|---|---|---|---|---|---|
| A | D | G | A | D | G |
| B | E | H | B | E | H |
| C | F | I | C | F | I |

## Activity 3

1. Cut out the top strip of the handout from Template A.
2. What slopes through A, or a copy of A, would be possible?
3. In a strip which has an infinite number of copies of the original square, what slopes would be possible through any point A?
4. What slopes through A would not lead to another point on the infinite strip?
5. With a partner, tape two strips together.
6. What slopes through A are now possible?

7. If there were an infinite number of infinite strips taped together, make a generalization about the slope of the line passing through A and any other point in this space.

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**Up:** *The Shape of Space Curriculum Materials*

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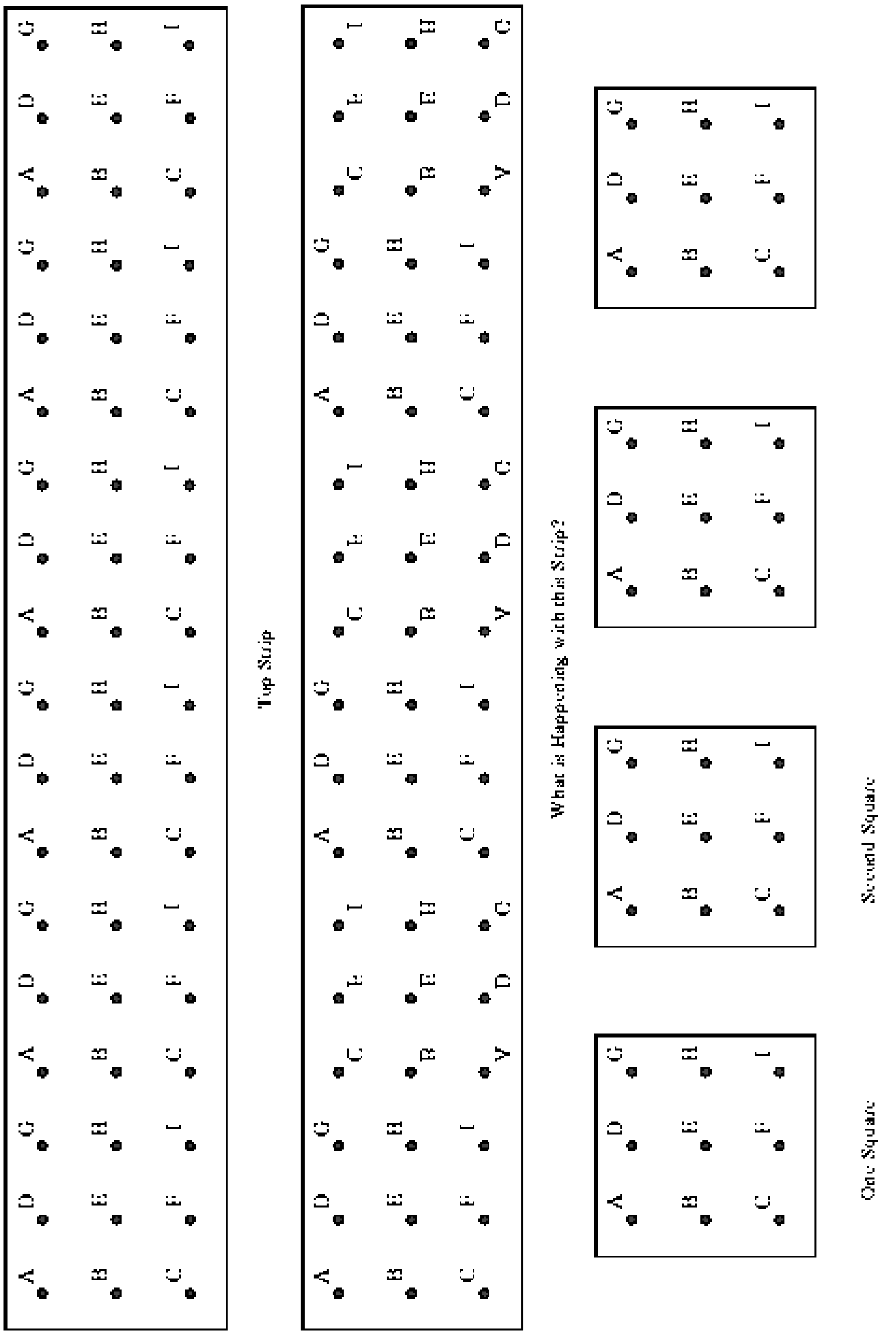
# Line-of-sight: Template A

*The Shape of Space Curriculum Materials: Geodesics*





# Symmetry and Tiling

**The Shape of Space Curriculum Materials**

## Symmetry and tiling in two dimensions

## Symmetry and tiling in three dimensions



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# Symmetry and 2-Dimensional Space

**The Shape of Space Curriculum Materials**

## Objectives

1. Students will be able to relate tiling to fundamental domains.

## Materials

1. *Symmetry and 2-Dimensional Space* Activity Sheet

## Introduction

Show the students a repeating pattern and ask what could be a representation of the pattern. Where do students find a repeating pattern in this design? What is the smallest stamp one could make which could generate this pattern? Is there more than one option for the making of this stamp?

Notice how a pattern is repeated. This is a *tiling*, a pattern that covers an area with no gaps and no overlaps. If we transfer this tiling to a transparency, it is possible to slide the transparency a certain distance until the transparency exactly matches the tiling everywhere.

In a periodic tiling of the plane you can identify a *fundamental domain.* A fundamental domain is the smallest tile with which you could cover the plane by using some combination of glides, rotations, reflections, and glide reflections. Different combinations of these operations produce different tilings.

## Activity

Hand out the activity sheet and have students work on finding fundamental domains on the patterns. Wrap up the activity by asking students the questions below.

## Questions

1. Are there any other options for the choice of fundamental domain? *[Yes for all but #4]*
2. Do all patterns have a fundamental domain? Why or why not? *[All on the activity sheet do, but not all patterns are periodic. Show an example.]*
3. Do all patterns have the same shape fundamental domain? Why or why not? *[No, parallelograms, hexagons and triangles are just a few of the possible shaped fundamental domains you could tile the plan with.]*

4. Do all fundamental domains of a tiling have the same area? *[yes]*

## Extensions

Have students look for fundamental domains in wallpaper samples.

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**Up:** *The Shape of Space Curriculum Materials: Symmetry and Tiling*

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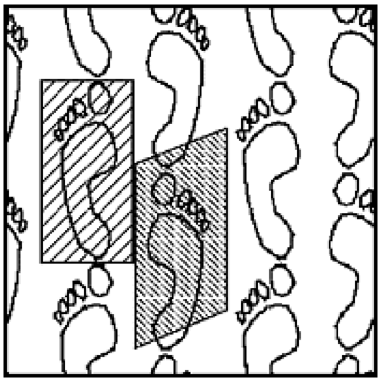
# Symmetry and 2-Dimensional Space

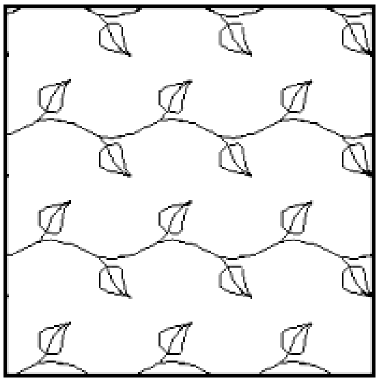
*The Shape of Space Curriculum Materials: Symmetry*

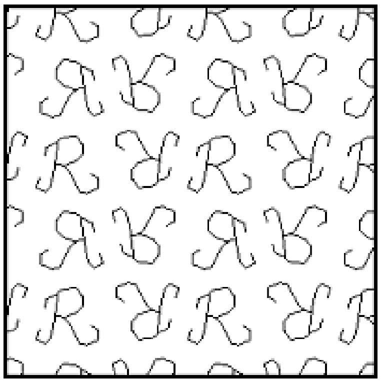

For each of the following patterns, look for a possible fundamental domain and mark your choice on the pattern. Some patterns have more than one possible fundamental domain.

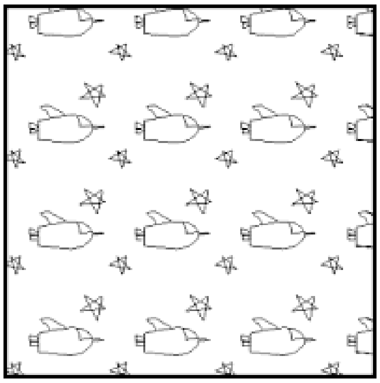
Two possible answers, each a different shape, have already been drawn on the first pattern. Each is a good answer, and there are many more as well.

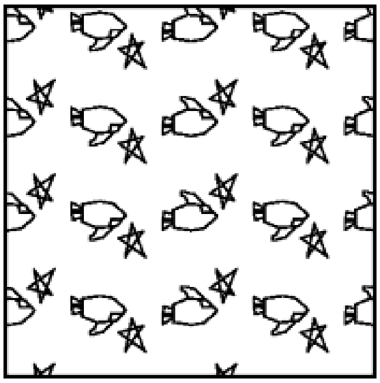
1.


2.


3.


4.


5.


6.


# Symmetry and 3-Dimensional Space

**The Shape of Space Curriculum Materials**

## Objective

Determine different symmetrical tilings of space using the cube as the fundamental domain.

## Materials

- 3-D Tiling Puzzle
- Clear Tape Or Glue
- Scissors
- 3/4" Diameter Color Coded Circular Labels

## Introduction

From previous activities with 2-dimensional surfaces you learned that gluing opposite edges together of a square in various combinations created surfaces which could be possible universes for a Flatlander. In addition, each gluing creates a symmetrical tiling that demonstrates what one would see as an observer on that surface. In other words, the gluing, the symmetry, and the surface all determine one another. This is also true of 3-dimensional space.

Euclidean 3-space is space that is unbounded and infinite. Can you think of a space that is unbounded and finite? Suppose our universe is a closed space in the same way a Flatland's universe is a closed surface. We can form a closed 3-dimensional universe by gluing opposite faces of a cube together in various combinations. This creates a symmetrical tiling of our motif in the images of it that would be seen from an intrinsic observer's point of view.

## Activity 1

Using an overhead transparency with the drawing from the 3-D Tiling Puzzle burned into it, assemble a cube with the figure suspended in the cube's interior. Placing color coded circular labels on the six faces of the cube enhance identifying the front, back, top, bottom, left, and right faces. Repeat the process to assemble eight cubes total. The cube will be the shape we will use as a fundamental domain for the tiling of 3-dimensional Euclidean space and the space ship will be the motif in space.

## Activity 2

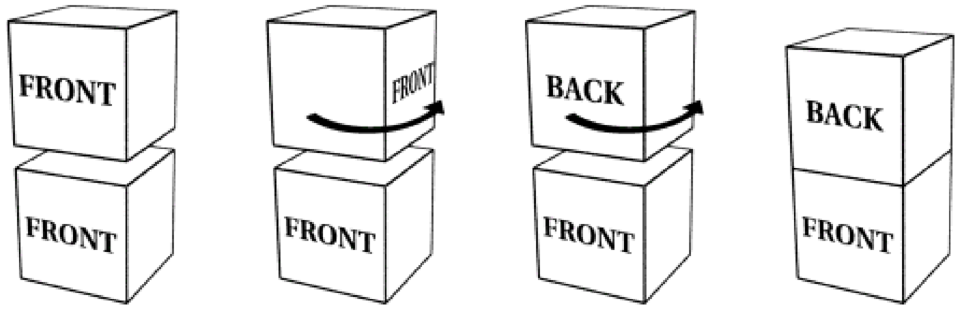
Create the following closed 3-dimensional universe and describe what an observer would see concerning the motif and its moves. Use the eight fundamental domains created in Activity 1 to help you visualize. Create different 3-D tilings with each the following rules:

1. Attach the front faces to back faces.
2. Attach the left faces to right faces.
3. Attach the top faces to bottom faces.
4. Do both 1 and 2.
5. Do both 1 and 3.
6. Do both 2 and 3.
7. Do all of 1, 2, and 3. (This is called a 3-torus closed universe.)

(To save time, just do 1, 4, and 7 above.)

# Activity 3

Create the following closed 3-dimensional universe by adding a "half twist" before gluing as shown in the diagram below. Describe what the an observer would see concerning the motif and its images.



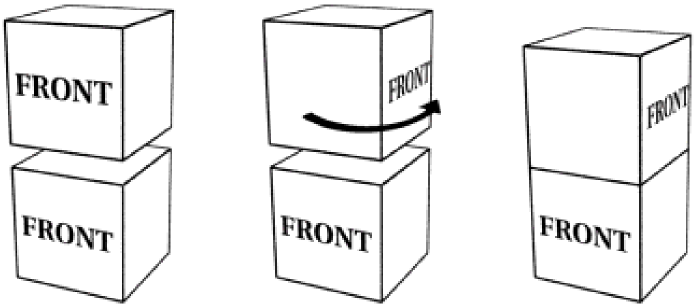

| Take the cube's top and bottom, | rotate one of them a quarter turn, | rotate it another a quarter turn, | and glue. |
|---|---|---|---|

1. Glue front faces to back faces with a half twist.
2. Glue left faces to right faces with a half twist.
3. Glue top faces to bottom faces with a half twist.
4. Do both 1 and 2.
5. Do both 1 and 3.
6. Do both 2 and 3.
7. Do both 1, 2, and 3.

(To save time, just do 1, 4, and 7 above.)

# Activity 4

Create the following closed 3-dimensional universe by adding a quarter turn clockwise, from the point of view of the figurine, before gluing as shown in the diagram below. Describe what an observer in this universe would see concerning the motif and its image. Which do not tile space symmetrically?




**Take the cube's top and bottom,** **rotate one of them a quarter turn,** **and glue.**

1. Glue front faces to back face with a quarter turn.
2. Glue left faces to right face with a quarter turn.
3. Glue top faces to bottom face with a quarter turn.
4. Combine 1 and 2.
5. Combine 1 and 3.
6. Combine 2 and 3.
7. Combine 1, 2, and 3.

(To save time, just do 1, 4, and 7 above.)

# Questions

1. Does gluing two opposite faces with a three quarter turn clockwise, from the point of view of the figurine, create a closed 3-dimensional universe that is the same as one created from a quarter turn?
2. What are other ways of gluing opposite faces together?

# Extensions

As a classroom activity, choose any combination of gluing faces and place the cubes from the entire class to tile space in a single stack. Each student takes a turn placing a cube in the stack with the correct orientation. Each student must verify whether the placing of the cube is consistent with the tiling in all directions. What other regular polyhedra will tessellate 3-dimensional space?

**Up:** *The Shape of Space Curriculum Materials: Symmetry and Tiling*

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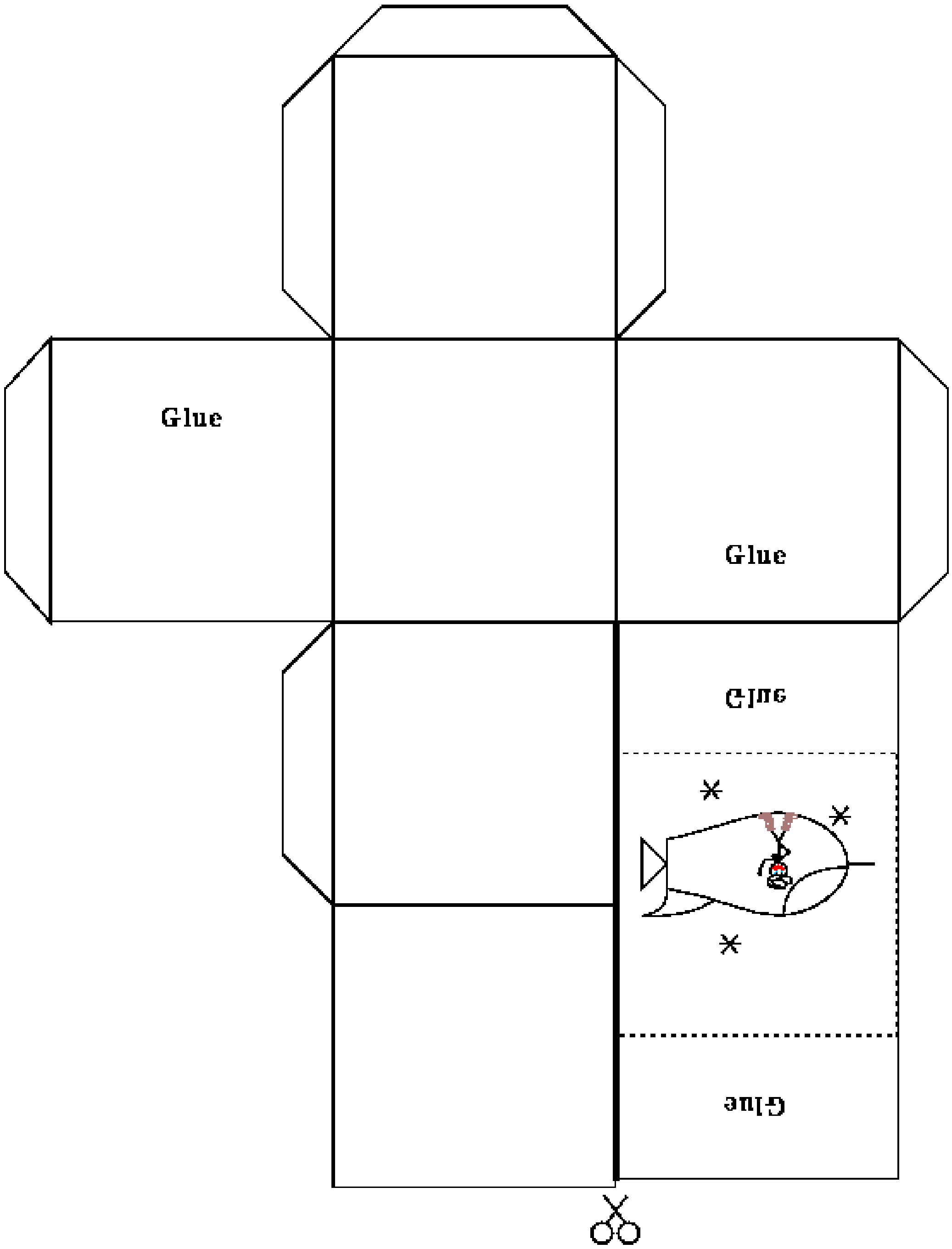
# Symmetry and 3-Dimensional Space

## Template M

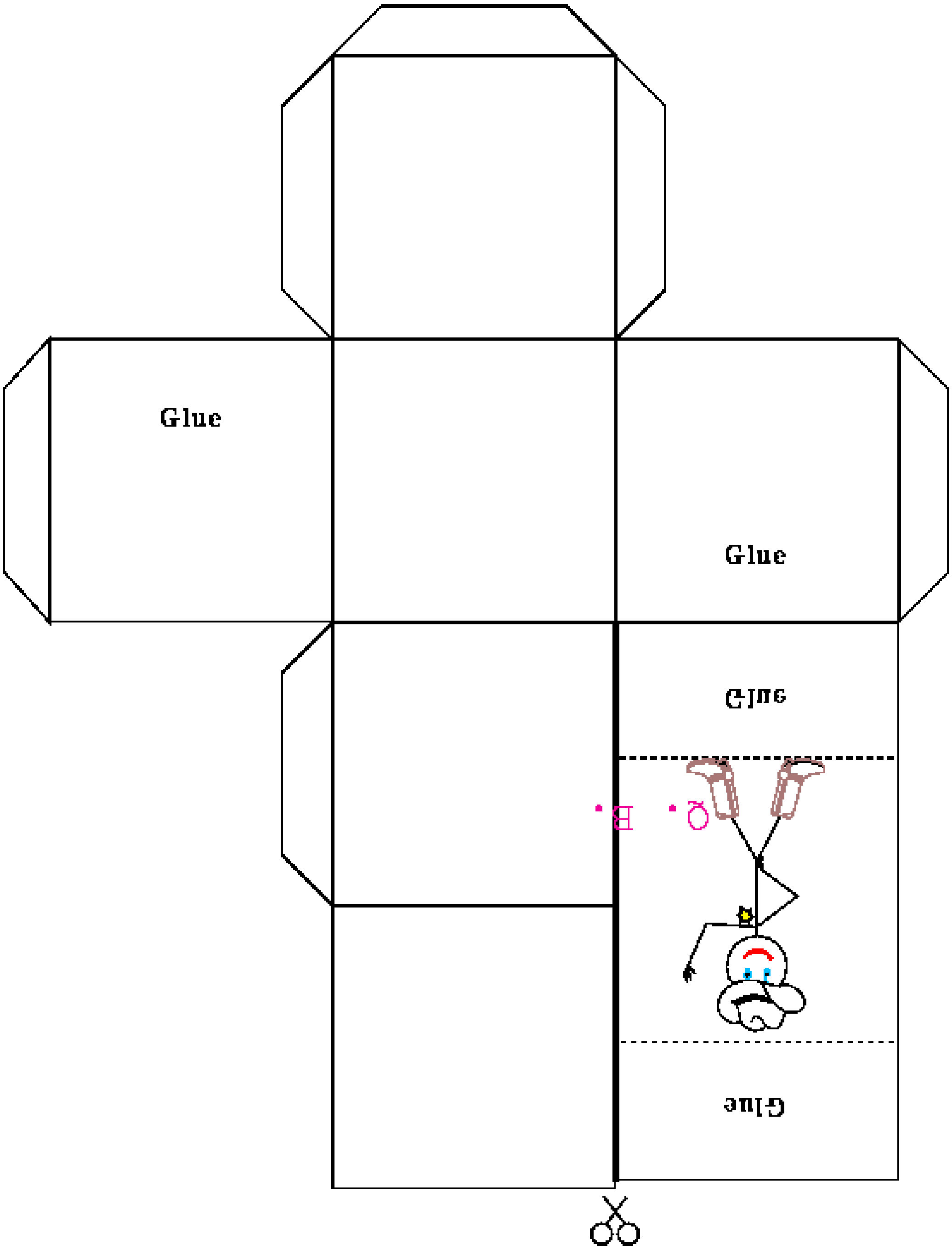
*The Shape of Space Curriculum Materials: Symmetry*


Glue
Glue
Glue
Glue

# Symmetry and 3-Dimensional Space

## Template N

*The Shape of Space Curriculum Materials: Symmetry*


Glue
Glue
Glue
Q. B.
Glue

# The Shape of Our Universe

**The Shape of Space Curriculum Materials**

## Objectives

- Students will be able to distinguish between bounded and boundless spaces.
- Students will be able to distinguish between finite and infinite spaces.
- Students will apply concepts of space learned from previous activities to our three-dimensional universe.

## Materials

- The Shape of Our Universe Activity Sheet
- The Shape of Space video

## Activity

Go over the meanings of *boundary*, *boundless*, *bounded*, *infinite* and *finite*, giving examples of each. Have students work in groups or pairs on the activity sheet.

When students have completed the worksheet, discuss how this all applies to the three-dimensional universe we live in. Let students express their ideas of what the universe might be like. Do they think it's boundless? Is there an edge of the universe? Does space go on forever? Could you go forever in a straight line and never see the same thing twice?

Show The Shape of Space video again as a final wrap-up.

## Answers

1. boundless: **c** (Klein bottle) and **e** (projective plane)
2. bounded in all directions: **b** (disc)
3. part bounded, part unbounded: **a** (annulus) and **d** (Möbius band)
4. finite:**all of the above**
5. 
    1. a line: **infinite**
    2. a line segment: **finite**
    3. a ray: **infinite in one direction, finite in the other**
    4. a circle: **finite**
6. 
    1. infinite and bounded in all directions: **No.**
    2. infinite and boundless in all directions: **Yes.**
    3. finite and bounded in all directions: **Yes. (e.g. a disc)**

4. finite and boundless in all directions: **Yes. (e.g. a torus)**

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**Up:** *The Shape of Space Curriculum Materials*

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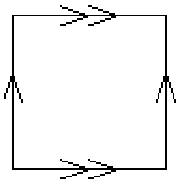
# The Shape of Our Universe

*The Shape of Space Curriculum Materials: Symmetry*

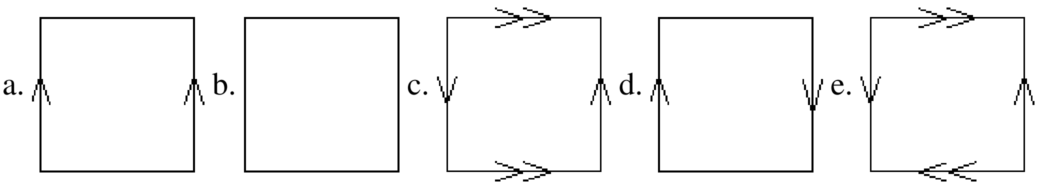


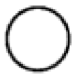
If the shape of the flatlander's universe is a torus, then everything in their universe can be represented in a gluing diagram like the one at the right.



There are no boundaries in this space; the flatlander can travel forever in any direction. But that does not mean that there is an infinite amount of space and matter in their universe. Travelling in a straight line forever, the flatlander would pass the same stars again and again. Even though their space is *boundless*, it is *finite* (not infinite).

1. Which of the spaces below are boundless (i.e. a flatlander could travel forever in any direction)? ___________________
2. Which have boundaries in all directions? ___________________
3. Which have boundaries in some directions but not others? ___________________
4. Which of the spaces are finite? _____________________




5. Which of the following are finite and which are infinite?
   a.  a line? ________________
   b.  a line segment? ________________
   c.  a ray? ________________
   d.  a circle? ________________

6. Is it possible for a space to be
   a. infinite and bounded in all directions? _____________
   b. infinite and boundless in all directions? _____________
   c. finite and bounded in all directions? _____________
   d. finite and boundless in all directions? _____________

What does this all mean about the three-dimensional universe we live in? It's possible that our universe is both finite and boundless. The following is from The Shape of Space video:

> *What seems to be a star in a distant galaxy could be our own sun. The light we receive from it could be light which left the sun billions of years ago, travelled around the universe, and is just now completing its trip. If we can someday find a pattern in the arrangement of the galaxies, then we will know the true shape of space.*

# Dimensions - Glossary

## The Shape of Space Curriculum Materials

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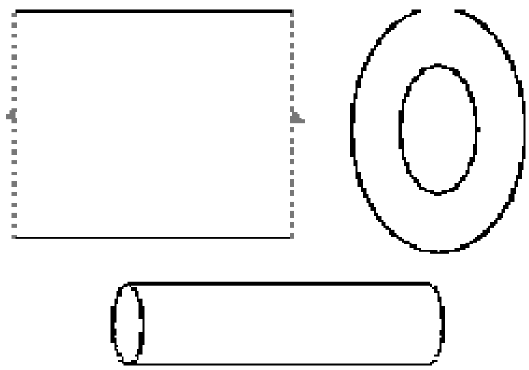
**annulus**

a tube



*Each of the above is an annulus: a gluing diagram of an annulus, a ring, and a tube.*

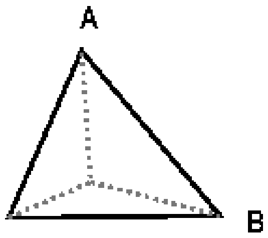
**boundary**

the border of a region

*The above box has one boundary that runs along all four sides.*

**edge**

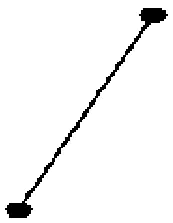
the border between two connected faces of a polyhedron



*In the above picture, segment AB is an edge of the tetrahedron.*

**finite**

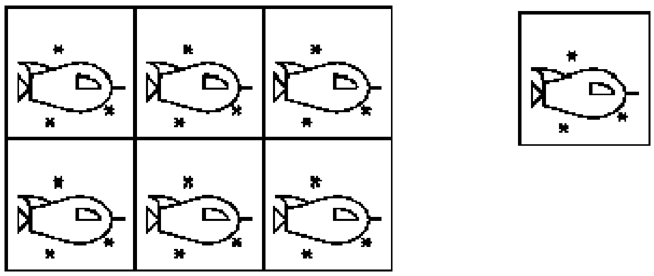
not infinite



*The above segment is finite. It does not extend forever in any direction.*

**fundamental domain**

the smallest part of a pattern which can be repeated to form the entire pattern



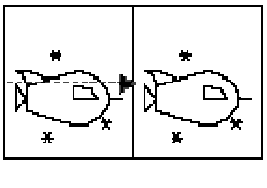
*In the above picture, the small square on the right with its contents is the fundamental doman for the tiling on the left.*

**geometry**

the study of the measurements of lengths, angles, area, and volumes; the study of the relationships between points, lines, angles, surfaces, and solids. Some of these change when a surface is deformed.

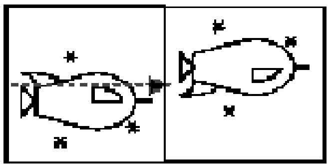
**glide**

translation; the transformation of an image by a distance and a direction



*The square on the left is the orignal square, called the preimage. The image to its right is formed* *by moving the preimage in the direction of the arrow. The total distance which the image is moved is equal to the length of the arrow. Note that the arrow is there for demonstrative purposes only; it is not part of the preimage.*

**glide-reflection**
a composition of a glide and a reflection; a translation and a reflection of an object

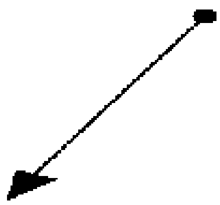


*Here, the preimage on the left is first reflected across the arrow and then a glide is applied to it (see glide), giving the image to the right.*

**gluing diagram**
a diagram with connected edges marked with an arrow for gluing.

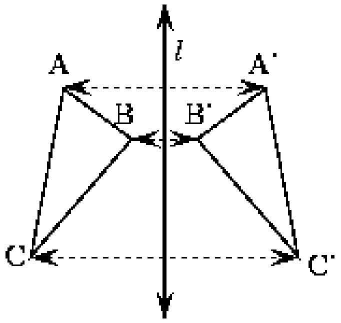
*This is a gluing diagram of a Mobius strip. If one were to cut it out,stretch it, and glue the sides together so that the arrows lined up, it would look like the representation of a Mobius strip in three dimensional space. This Mobius strip diagram has the same intrinsic topology of a Mobis strip placed in three dimensional space.*

**infinite**
without end



*The ray is infinite in the direction of the arrow.*

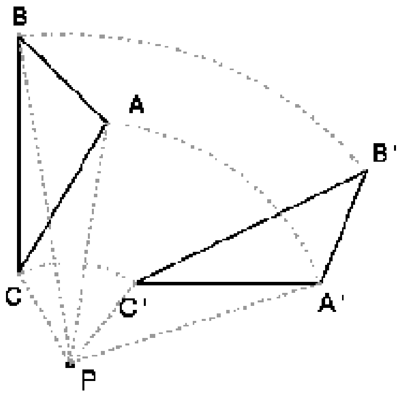
**reflection**
mirror image




*Triangle ABC is reflected about line l to make triangle A'B'C'.*

**rotation**

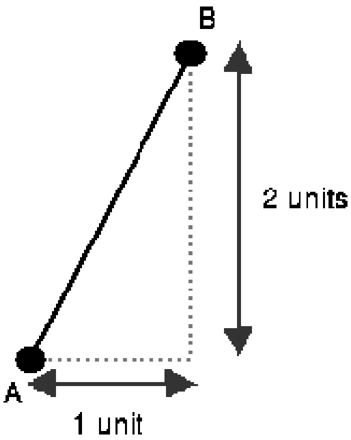
a spin around a certain point, called the center of rotation




*Triangle ABC is rotated (45 degrees) about point P to make triangle A'B'C'*

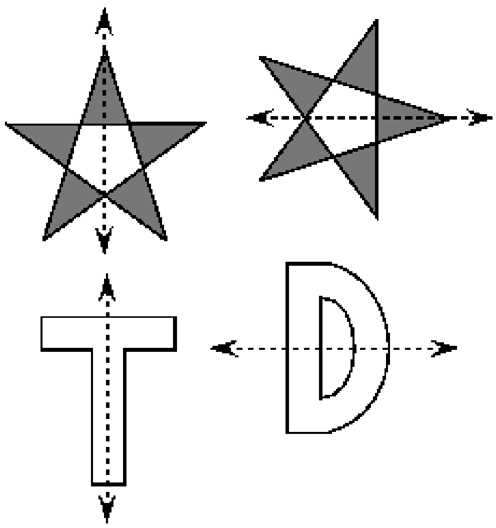
**slope**

the upward or downward slant of a line or the inclination of the slant of a line



*Segment AB extends up two units for every unit to the right that it extends. Thus, it has slope 2/1.*

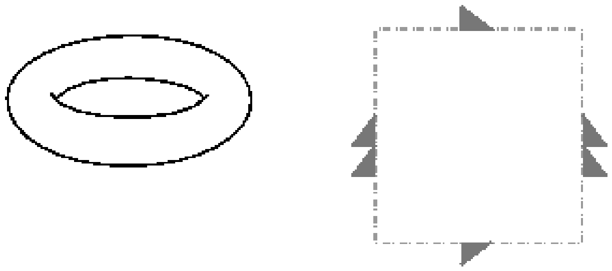
**symmetrical**
correspondence in size, shape, and relative position



*Each of the above shapes is symmetrical since the part of the shape on one side of the arrow is the same as the part of the shape on the other side of the arrow except for its position.*

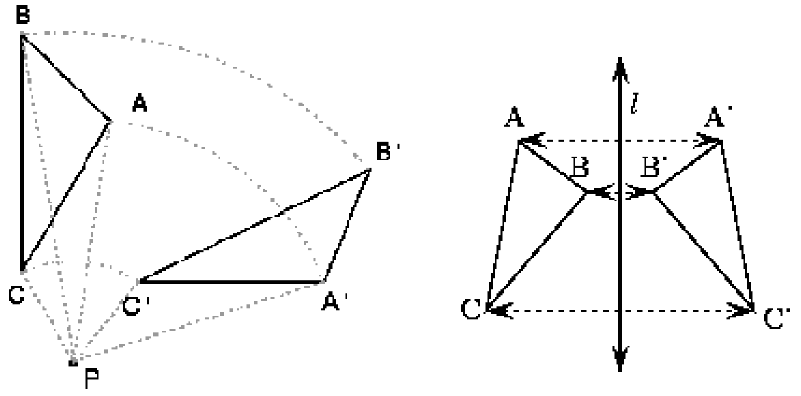
**topology**
the study of surfaces; the study of properties which do not change under certain transformations

**torus**
the surface of a donut; can also be represented by a flat gluing diagram



*Above is a doughnut-shaped torus and a torus gluing diagram. They are both intrinsically topologically a torus.*

**transformation**
a change like a glide, rotation, reflection, or glide-reflection




*The above rotation and reflection are examples of transformations*