Chapter 3

Tutorials

This manual is available on-line at the URL

`http://www.geom.umn.edu/locate/pisces/`

The on-line version may be more current, and may contain features not contained in the printed version. For example an the on-line version contains an interactive interface to Pisces via the World-Wide Web.

3.1 Getting Started: The Pisces Logo

In this section we describe how to compute the zero set of the default Pisces model. If you have never run Pisces, you may wish to read the section of documentation that describes how to launch Pisces.)

Upon beginning a new Pisces session, the main panel appears. Note that the default model is the Pisces Function, which means that this is the function for which Pisces will attempt to compute a level set.

3.1.1 Computing a Level Set

- Open the control panel for the Adaptive Grid Algorithm by selecting the first entry on the **Algorithms** menu.

- Press the **Go** button on the algorithm's control panel in order to request that Pisces use the adaptive grid algorithm (with the default parameters) to compute the level set of the current model.

- Open a View Window by selecting the first entry on the **Output** menu. The window should contain the Pisces logo, which is actually the level set of an 11th degree polynomial in two variables.

3.1.2 Manually Changing the Viewport

Edit the bounds of the view window's viewport to *[-5,5]x[-5,5]* in order to change the way that the Pisces logo appears on your screen. (Hint: After changing the minimum value of the x variable to read *-5*, `<TAB>` to the next field. When you have changed the entire viewport, press `<RETURN>` to submit the changes.)

3.1.3 Zooming with the Mouse

To zoom in on the Pisces logo, position the cursor of your mouse at the upper left corner of the logo (just above the tail of the "upper" fish). Hold down the rightmost mouse button while you drag out a rectangle that contains the lower right corner of the logo. The "rubber-band box" should encompass most or all of the Pisces logo. Release the right mouse button and select **Actions->Zoom in** in order to scale everything that was inside the rubber-band box until it fills the viewport.

3.1.4 Restoring the Default Viewport

Select the menu item **Actions->BBox->Viewport** in order to restore the viewport to the domain *[-2.2, 2.2]x[-1.2,1.2]*.

3.1.5 Modifying Model Parameters

Under the **Settings** menu on the main panel, choose the first item labeled **Model**. This action will create a new panel called the Model Panel. This panel enables the user to control each model's parameters, domain, and range. To see how this works, change the value of the parameter labeled **eyerad** to read *0.3*. Compute the level set for this new function by pressing the **Go** button on the control panel for the Adaptive Grid Algorithm. The View Window should display a new image of two fish with big eyes. Because the fish eyes are almost touching the fish's "forehead," this function is nearly singular, and so the level set in the vicinity of the forehead will look as if the fish have been attacked by some kind of necrotic fungus.

3.1.6 Modifying Algorithm Parameters

To cure the Pisces logo of its "scale rot," change the parameter **Max_Depth** in the Adaptive Grid Control Panel so that the parameter has the value *15*. Recompute the level set by pressing the **Go** button. After a few seconds, the fish will look much improved.

3.1.7 Printing

To print the big-eyed Pisces logo, open the Postscript Panel located on the **Output** menu of the Main Panel. On this panel, press the **Print** button to create a postscript file (called `pisces.ps`, by default). The file will appear in the directory from which you invoked Pisces, and may be sent to any postscript-compatible printer. On UNIX workstations, this is accomplished with the `lpr` command.

3.1.8 Ending the Session

If desired, change other parameters in the Model Panel and compute the related level set to get a feel for how the level set changes with the parameters.

Similarly, change parameters that control the Adaptive Grid algorithm to see how each parameter affects the computation of the level set.

When you are finished, end this Pisces session by selecting **Quit** from the **File** menu on the Main Panel.

3.2 A Family of Circles

This section of the tutorial assumes that you have completed the section on computing level sets of the default model. In this section we describe how to animate the computation of a level set for a family of functions.

Launch Pisces and open the Model Panel, the Adaptive Grid Control Panel, and the View Window.

3.2.1 Changing Models

We want to compute level sets that are circles, so we first need to choose a model that has such levels. Select the Cubic 2D model from the **Model** menu on the Main Panel. The Model Panel should update to reflect that you have changed models.

In the Model Panel, set all parameters equal to zero except for `coef_x^2` and `coef_y^2` which should be set to 1. The current model is now set to be *x^2 + y^2*. We want to compute the level set of 1, therefore change the range value to *c=1*.

Lastly, we want to compute the level set on the domain *[-2,2]x[-2,2]*, so modify the entries that determine the bounding box in which Pisces will compute a level set. (Hint: Use <`TAB`> to move from field to field when inputting data.)

Finally, press the **Go** button on the Adaptive Grid Control Panel to compute the unit circle *x^2+y^2=1*. (If you don't see anything in the View Window, make sure you have set *c=1* in the Model Panel.) The circle will look elliptical because the viewport of the View Window is not square. Correct this by telling Pisces to use the current bounding box as the current viewport. To do this, select **BBox->Viewport** from the **Actions** menu of the View Window.

3.2.2 Animation

Now we will compute a family of level sets as a parameter is varied. Open the Animation Panel by selecting the **Animate** item from the **Utilities** menu on the Main Panel. On the Animation Panel, find the menu labeled **Parameter to Animate** and select **Model.c** from that menu. Set the values of the **Starting Value** to 0.5, the **Ending Value** to 4.0, the **Number of Steps** to 7, and then press the **Go** button on the Animation Panel. You should see 7 circles appear on the View Window, each circle replacing the previous circle.

3.2.3 Superimposing Data

Suppose you wanted to create a contour plot of the function *x^2+y^2* or, equivalently, you want to simultaneously look at a family of level sets for the same model. To do this, open the Output Panel by selecting the **Settings->Output** menu option from the Main Panel. Toggle the **New Geometry** option so that it is set to **adds to last**. Now clear all of the current Pisces data by selecting the **Output->Erase all Graphics** item from the Main Panel. Once again press **Go** on the Animation Panel and see a family of circles superimposed in the same View Window.

3.2.4 Viewing 3D Data

This portion of the tutorial is only valid for those installations that have hooked up Geomview to interface with Pisces. To begin, clear all Pisces graphics by selecting the **Output->Erase all Graphics** item from the Main Panel.

From the **Output** menu on the Main Panel, select the **Geomview** option. After a short wait, a Geomview session should begin. (This may not happen if you are running Pisces remotely. For example, if you are running Pisces on an SGI computer but displaying the results on a Sun.)

Once Geomview has been launched, once again press the **Go** button on the Animation Panel. As the level sets are computed, they will be simultaneously plotted in Geomview. After the level sets appear, you can use Geomview to zoom out and rotate the levels, which sweep out an approximation to the graph of *x^2+y^2*.

3.2.5 Ending the Session

If desired, use the animation panel to animate other parameters of the current model.

When you are finished, end this Pisces session by selecting **Quit** from the **File** menu on the Main Panel. If Geomview is running, you will need to quit that application separately.

3.3 Conic Sections

This section of the tutorial assumes that you have completed the section on computing level sets of the default model. In this section we describe how to animate the level sets of a two-parameter family of functions in order to see the transition between various conic sections (ellipses, hyperbolas, parabolas, etc.)

We will also use a new algorithm, called the Predictor-Corrector algorithm.

Start a new session of Pisces, and open the Model Panel, the Predictor-Corrector Control Panel, and the View Window.

Select the Cubic 2D model from the **Model** menu on the Main Panel.

We will study the two-parameter family of quadratic polynomials given by
x^2 + ay + by^2
as the parameters a and b are varied.

3.3.1 Two Parameter Animation

Let's compute the level set of *x^2 + a y + b y^2* as the parameters a and b both vary. One simple way to let the parameters vary is to vary them along a circle in parameter space. A parametrization of a circle in parameter space is *a(t) = cos(t), b(t) = sin(t)*, so let's animate the level sets as a and b vary along this path.

Open the Parametric Animation Panel by selecting **Parametric Animate** from the **Utilities** menu of the Main Panel. Change the very first entry marked **Steps** so that we will take 16 animation steps. About halfway down the panel are two menus that allow the user to select the parameters to be animated. Select **Model.coef_y** and **Model.coef_y^2**. Under the menu labeled **Algorithm**, select **Predcorr**, press the button marked **Show Parametric Path** so that the pth in paramter space is shown, and then press the button marked **Go**.

What you will see is a parabola transform into a series of ellipses, another parabola, hyperbolas, and intersecting lines.

3.3.2 Computing Parameters Corresponding to Singular Curves

In the previous animation, there were several level sets that were *singular* in the sense that there was a point on the curve for which the gradient of the model vanished. The parameter values for which a level curve is singular will typically form a hypersurface in parameter space (in this case, a curve in *(a,b)*-space). This hypersurface is often (but not always) indicative of a change in the topology of the level curve.

To compute the set of parameter values corresponding to singular curves,

Create a Derived Model

Let *f* be the current model. Then singular curves correspond to the zero set of the function *F=(f, df/dx, df/dy)*. This is a new model which is *derived* from the model *f*.

To create this model, select **Derived Model** from the **Settings** menu on the Main Panel. The Derived Models Panel will appear. On this panel, select the menu item labeled **Planar Singularity**, then press the **Update** button that appears.

Inflate the Model Domain

At this point, we have a model with a two-dimensional domain, *(x,y)*, and a three-dimensional range, *(f, df/dx, df/dy)*. In order to use Pisces, however, the dimension of the domain must be greater than the dimensions of the range. The solution of this problem is to *inflate* the domain by two parameters. At the bottom of the Model Panel, press the **Permute** button to bring up the Permutation Panel. Use the Permutation Panel to tell Pisces that the variables **coef_y** and **coef_y^2** should be added to the domain, rather than held fixed as parameters. There are now four variables selected to be in the domain, so press the **OK** button in order to dismiss the Permutation Panel and apply the permutation.

Compute the Zero Set of the Derived Model

Press the **Go** button in the Predictor Corrector Control Panel. Pisces will compute a curve in *R^4*; the View Window displays the projection of this curve into *(x,y)*-space. We are more concerned with the projection of this curve into parameter space, so use the **Hor** and **Ver** menus of the View Window in order to change the View Window so that the solution curve is projected onto a vertical line in *(coef_y, coef_y^2)*-space. It is easy to show analytically that the set of singular curves for the family *x^2 + a y + b y^2* occurs for *a=0*, so Pisces has correctly computed the singular set.

3.3.3 Ending the Session

When you are finished, end this Pisces session by selecting **Quit** from the **File** menu on the Main Panel.

3.4 The Unfolding of a Cusp

This section of the tutorial assumes that you have completed the section on computing level sets of the default model. We also assume that you have completed the previous section on Conic Sections. In this section we animate the level sets of a two-parameter family of functions (called the "unfolding of a cusp") and compute the singular set for this family.

Start a new session of Pisces, and open the Model Panel, the Predictor-Corrector Control Panel, and the View Window.

3.4.1 Setting up the Model

Select the Cubic 2D model from the **Model** menu on the Main Panel.

We will study the two-parameter family of quadratic polynomials given by
x^3 - y^2 + ax + b
as the parameters a and b are varied. To begin, set all of the parameters in the Model Panel to zero, except for the parameters **coef_x^3** (set equal to 1) and **coef_y^2** (set equal to -1).

3.4.2 Computing an Initial Level Set

We wil compute the level set of *x^3 - y^2* using the Predictor-Corrector Algorithm, paying particular attention to the presence of singularities in the model.

Change the parameter **Singular_Points** on the Predictor-Corrector Control Panel to read 1. This sets up a *1 x 1* array of initial points which Pisces will use to attempt to detect singularities in the current model. Pressing the **Go** button on the Predictor-Corrector Control Panel will cause Pisces to detect the singularity at the origin, to locally represent the structure of the singularity, and then to trace the remainder of the level set.

3.4.3 Two Parameter Animation

The mathematical question behind this section of the tutorial is "How do small perturbations of a function affect level sets for that function?" In general, small perturbations will not change the topology of a nonsingular level set, whereas small perturbations may change the topology of singular levels. In this section we will perturb the function whose level set is a cusp in order to gain insight into how the level set changes under perturbation.

We will once again use two-parameter animation and let parameters vary along a circle in parameter space. Open the Parametric Animation Panel by selecting **Parametric Animate** from the **Utilities** menu of the Main Panel. Change the very first entry marked **Steps** so that we will take 16 animation steps.

About halfway down the panel are two menus that allow the user to select the parameters to be animated. Select **Model.coef_x** and **Model.const_coef**. Under the menu labeled **Algorithm**, select **Predcorr**, then press the button marked **Go** .

What you will see is a single curve transform into a pair of curves, and then back again.

3.4.4 Computing Parameters Corresponding to Singular Curves

As in the previous section, we will compute a curve in *(x,y,a,b)*-space; each point on the curve corresponds to a function (indexed by *(a,b)*) whose level set is singular (at the point *(x,y)*.

As in the previous problem:

Create a Derived Model

Select **Derived Model** from the **Settings** menu on the Main Panel. The Derived Models Panel will appear. On this panel, select the menu item labeled **Planar Singularity**, then press the **Update** button that appears.

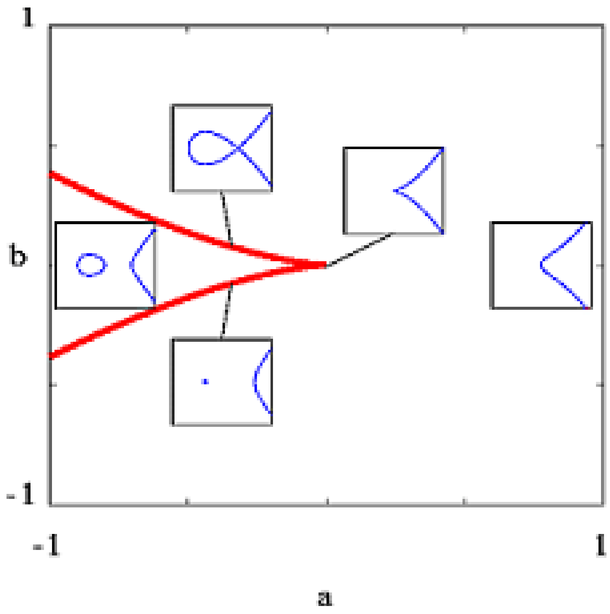
Inflate the Model Domain

At the bottom of the Model Panel, press the **Permute** button to bring up the Permutation Panel. Use the Permutation Panel to add the variables **coef_x** and **const_coef** to the domain. (Don't forget to press the **OK** button in order to dismiss the Permutation Panel and apply the permutation.)

Compute the Zero Set of the Derived Model

Press the **Go** button in the Predictor Corrector Control Panel. Change the **Hor** and **Ver** coordinates of the View Window in order to project the solution curve onto *(a,b)*-space. This curve should look like a cusp.

The curve separates the parameter space into two regions. For level curves indexed by parameters in the region containing *(a,b)=(1,0)*, the topology of the level set is that of a line; for level curves indexed by parameters in the region containing *(a,b)=(-1,0)* are topologically equivalent to a circle and a line. For parameter values on the separating curve, the corresponding level set is singular. This information is summarized in the image below.



Level sets for Perturbations of the Cusp.

3.4.5 Ending the Session

When you are finished, end this Pisces session by selecting **Quit** from the **File** menu on the Main Panel.

3.5 The Pitchfork Bifurcation

In this section we compute a bifurcation diagram for a family of one-dimensional differential equations. In particular, we compute the location of equilibria for a one-parameter family, and compute the saddle-node bifurcations for this family.

Start a new session of Pisces, and open the Model Panel, the Predictor-Corrector Control Panel, and the View Window.

3.5.1 Setting up the Model

Select the DS Pitchfork model from the **Model** menu on the Main Panel.

We will study the three-parameter family of vector fields on the line given by
x^3 - mu x + alpha + beta mu
as the parameters *mu, alpha* and *beta* are varied.

3.5.2 Computing a Locus of Equilibria

We wil compute the locus of equilibria for this vector field in *(x,mu)*-space. To begin, we must tell Pisces to compute equilibria and also to inflate the domain by the parameter *mu*.

Create a Derived Model

Select **Derived Model** from the **Settings** menu on the Main Panel. The Derived Models Panel will appear. On this panel, select the menu item labeled **Fixed Points ODE**, then press the **Update** button that appears.

Inflate the Model Domain

At the bottom of the Model Panel, press the **Permute** button to bring up the Permutation Panel. Use the Permutation Panel to add the variable **mu** to the domain. Press the **OK** button in order to apply the permutation.

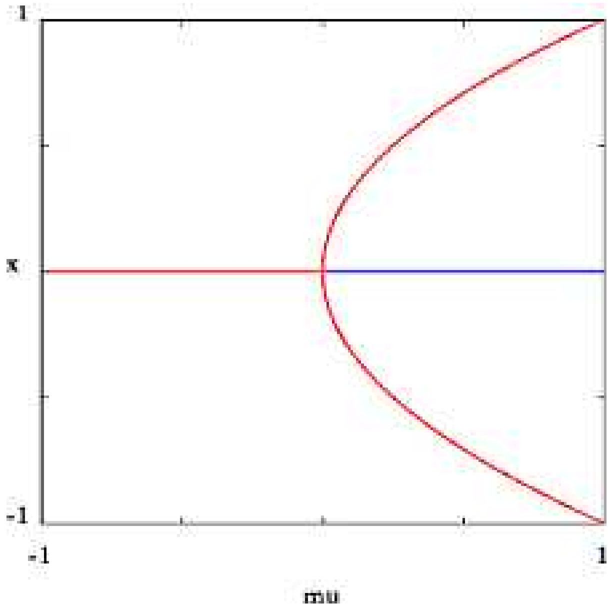
Compute the Zero Set of the Derived Model

Change the following parameters on the Predictor-Corrector Control Panel:

- **Uniform_Points** = 0
- **Boundary_Points** = 3
- **Turning_Points** = 3
- **Singular_Points** = 1

Now press Predictor-Corrector's **Go** button. Change the **Hor** and **Ver** coordinates of the View Window in order to project the solution curve onto *(mu,x)*-space. This curve should look like the pitchfork below (though colors may be different).

For *mu<0*, the vector field has only one (unstable) equilibria; for *mu>0*, there are three equilibria (two unstable; one stable).



A Locus of Equilibria.

3.5.3 One-Parameter Animation of a Bifurcation Diagram

If the parameter *beta* is nonzero, the *x -> (-x)* symmetry in this problem is broken. This symmetry-breaking bifurcation is discussed in Chapter 1 of M.

Golubitsky and D. Schaeffer, *Singularities and Groups in Bifurcation Theory, I*, Springer, 1985.

To view the symmetry-breaking bifurcation, first make sure that the Predictor-Corrector parameters are set as indicated above. Open the Animation Panel by selecting **Animate** from the **Utilities** menu of the Main Panel. Animate the parameter **Model.beta** in eleven (11) steps within the interval *[-0.3, 0.3]*, using the **Predcorr** algorithm.

What you will see is a pair of nonsingular curves transform into a singular (symmetric) curve and then back into nonsingular curves.

3.5.4 Computing Saddle-Node Bifurcations

We will compute a curve in *(x,mu, beta)*-space corresponding to saddle-node bifurcations. Geometrically, these are points on the curves of equilibria for which the curve is turning with respect to the parameter *mu*.

As in the previous problem:

Create a Derived Model

On the Derived Models Panel, select the menu item labeled **Saddle Node ODE**, then press the **Update** button that appears.

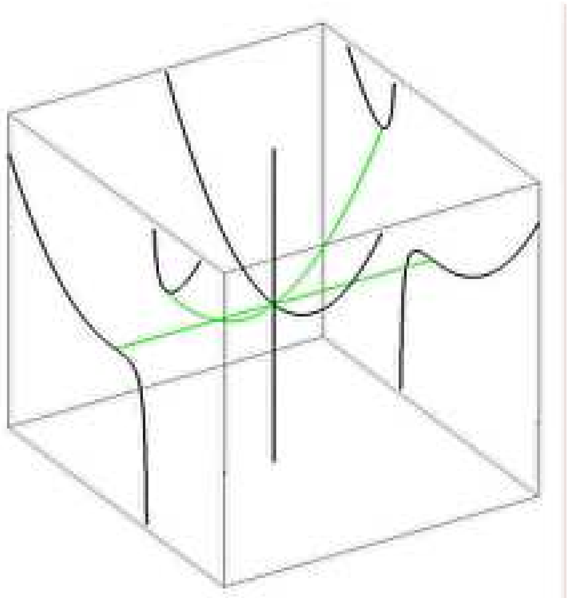
Inflate the Model Domain

At the bottom of the Model Panel, press the **Permute** button and use the Permutation Panel to add the variable **beta** to the domain. Press the **OK** button to apply the permutation.

Compute the Zero Set of the Derived Model

Press the **Go** button in the Predictor Corrector Control Panel. Change the **Hor** and **Ver** coordinates of the View Window in order to project the saddle-node curve onto *(mu,beta)*-space. The saddle-node curve separates the parameter space into four regions. A saddle-node bifurcation occurs for parameter values on the curve.

If Geomview is installed on your system, you can view the saddle-node curve in *(x,mu,beta)*-space. Launch Geomview by selecting the **Geomview** item from the **Output** menu. When Geomview appears, it should show a three-dimensional image of the saddle-node curve. If it does not, press the **Close** button on the Main Panel and then reopen the connection via **Output->Geomview** in order to force Pisces to resend its current graphics buffer.



A Saddle Node curve in *(x,beta,mu)*-space. Three curves of equilibria for fixed *beta* are included to show that the saddle-node curve corresponds to turning points of the curve of equilibria.

3.5.5 Ending the Session

When you are finished, end this Pisces session by selecting **Quit** from the **File** menu on the Main Panel.