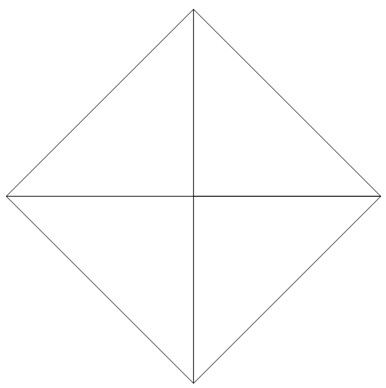
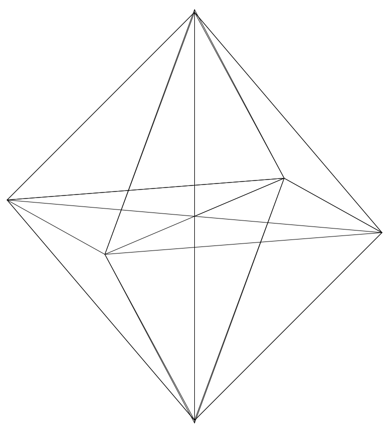
To the Students at the Geometry Center on April 29, 1995

Let's finish that problem we started to discuss at the end of the morning, about the volume of a ball in higher dimensions. We should be able to establish the formula for the 4-ball, and hopefully get enough information so we can make a good conjecture about the n-dimensional volume of a ball in n-dimensional space.

We can start with some estimates that don't even use calculus. A ball of radius R sits inside a cube of side length $2R$ so in general the volume $V_n(R)$ of an n-ball of radius R will be less than $(2R)^n = 2^n R^n$.

Certainly zero is a lower bound for the volume and we can do better by observing that the n-ball must contain the analogue of the diamond in the plane and the octahedron in 3-space, those being the smallest figures that contain the points at distance R along each of the coordinate axes. Since the vertices of this figure are the midpoints of the sides of the analogue of the cube, the figure is called the n-dimensional $cube - dual$. In the plane we get four triangles, each one-half the area of a subsquare of side length R in one of the quadrants, for a total area of $4(R^2)(1/2) = 2R^2$. In 3-space, we get eight triangular pyramids, one in each octant, each with volume $(1/3)(1/2)R^3$ for a total of $(2^3)(R^3)/(2\cdot 3)$. In general the n- dimensional volume of the n-dimensional cube dual will be $(2^n)(R^n)/n!$.

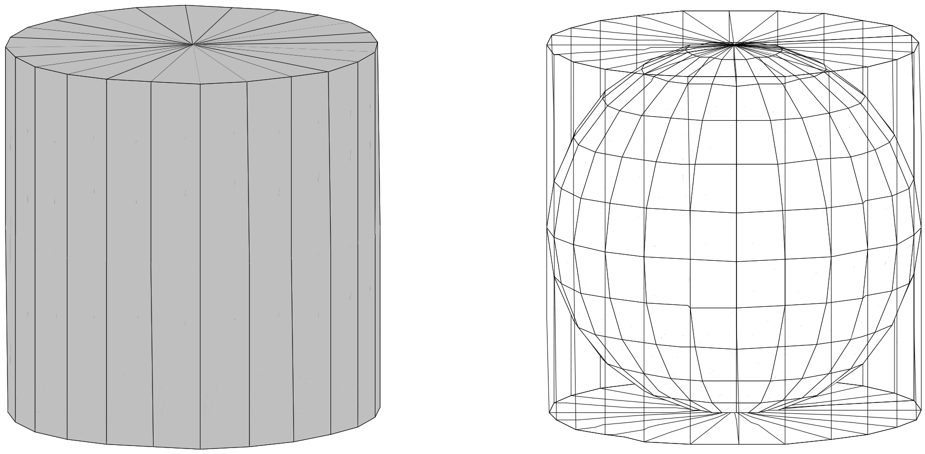




For low dimensions, this is a reasonably good lower bound: For $n = 2$, we have $2R^2 < \pi R^2 < 4R^2$, and, using the familiar formula, if $n = 3$, we have $(8/6)R^3 < (4/3)\pi R^3 < 8R^3$, not such a good approximation. It gets worse, since $(2^n)/n!$ will approach 0 as n goes to infinity. This is easy to see since from 4 onwards, at each step we multiply the numerator by 2 and the bottom by something at least twice that great, so the new expression is less than half the value at the previous step.

This leads to an interesting question. Once we do find an expression for the volume of the n-ball, what happens to the ratio of this volume to the volume $2^n R^n$ of the n-cube as n becomes arbitrarily large?

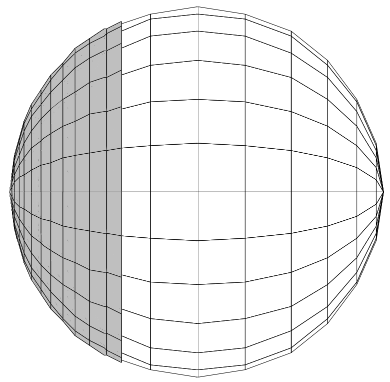
It is easier to get some better upper bounds, although good lower bounds are a bit harder to come by. In 3-space, the ball of radius R sits inside a solid cylinder of radius R and height $2R$, so $V_3(R) < (2R)(\pi R^2) = (2\pi)R^3$. As it happens, $V_3(R) = (4/3)\pi R^3$, so the volume of the 3-ball is two-thirds the volume of the circumscribing cylinder, a discovery that Archimedes found so significant that he instructed it to be inscribed on his tombstone.



We may think of the solid cylinder in 3-space as the "product" of a disc of radius R in the coordinate plane of the first two coordinates, and a segment of length $2R$ in the remaining coordinate line. In 4-space, we will have four independent coordinate directions, and the 4-ball will be contained in the "product" of a disc of radius R in the plane of the first two coordinates and another disc of radius R in the plane of the second two. Thus $V_4(R) < \pi^2$. As it happens, $V_4(R) = (1/2)(\pi^2)$ so the hypervolume of the 4-ball of radius R is exactly one-half the hypervolume of the "bicylinder" which is the product of two discs of radius R. It isn't clear who was the first mathematician who discovered this, but it was certainly known in the early part of the nineteenth century, when people first began to ask questions about the higher-dimensional measurements analogous to area in the plane and volume in 3-space.

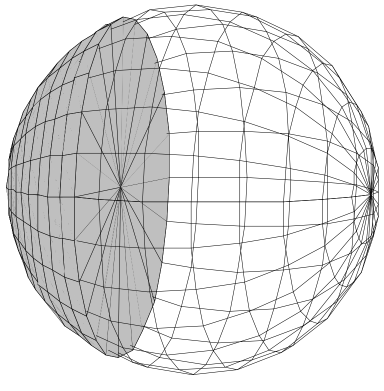
The basic insight that links volume in n-space with elementary calculus is that once we can find the derivative of a function of one variable, we can use techniques of integration to approximate the function, and sometimes we can compute it exactly. This is the content of the Fundamental Theorem of Calculus, the most powerful tool for calculating areas, volumes, and hypervolumes.

Using the slicing idea studied in *Flatland*, we can express the area of a disc of radius R by moving a vertical line across the disc and keeping track of the length of the intersection, starting with length 0 at position $-R$, then increasing to $2R$ at position 0, thereafter decreasing to 0 at position R. By the Pythagorean theorem, the length of the segment at postion x is $2\sqrt{R^2 - x^2}$, and the integral of this quantity with respect to x from $-R$ to R will give the area of the disc.



Note that either by using a "change of variable" formula, or by observing the effect of scaling on approximating rectangles, we can see that the integral of $2\sqrt{R^2 - x^2}$ from $-R$ to R with respect to x is R^2 times the integral of $2\sqrt{1-t^2}$ from -1 to 1 with respect to t, and this last quantity, the area of the unit disc, is defined to be π. Thus $\int_{-R}^{R} 2\sqrt{R^2 - x^2}\, dx$ $= R^2 \int_{-1}^{1} 2\sqrt{1-t^2}\, dt$ and $V_2(R) = \pi R^2$.

In 3-space, we can compute the volume of a ball of radius R by slicing it by planes perpendicular to the x-axis and keeping track of the areas of the slices as x runs from $-R$ to R. By the Pythagorean theorem once again, the slice of the sphere at position x is a disc of radius $\sqrt{R^2 - x^2}$, with area area $\pi\sqrt{R^2 - x^2}^2 = \pi(R^2 - x^2)$. The antiderivative of this expression with respect to x is $\pi(R^2 x - x^3/3)$ so $\int_{-R}^{R} \pi(R^2 - x^2)\, dx = \pi((2/3)R^3 - (2/3)(-R)^3) = (4/3)\pi R^3$, the familiar formula for the volume of a 3-ball.

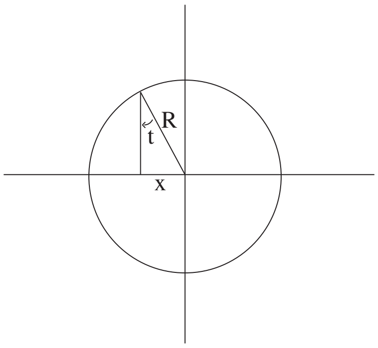


In 4-space, as we slice a 4-ball by 3-dimensional *hyperplanes* perpendicular to the x-axis, we obtain 3-dimensional balls as the slices. We may then compute the hypervolume of the 4-ball of radius R by keeping track of the volumes of these slices at position x as x runs from $-R$ to R. The volume of the slice at position x is $(4/3)\pi\sqrt{R^2 - x^2}^3$ and our task is to integrate this from $-R$ to R with respect to x.

We can use this expression to get an upper bound on $V_4(R)$ since $(4/3)\pi(\sqrt{R^2 - x^2})^3 = (4/3)\pi(R^2 - x^2)\sqrt{R^2 - x^2} = (4/3)\pi(R^2)\sqrt{R^2 - x^2} - (4/3)\pi(x^2)\sqrt{R^2 - x^2}$. This gives the integral of the difference of two expressions, so the integral of the first will be larger than $V_4(R)$. But $\int_{-R}^{R}(4/3)\pi(R^2)\sqrt{R^2 - x^2}\,dx = 2/3\pi R^2$ multiplied by the area of the 2-disc of radius R, so $V_4(R) < (2/3)\pi R^2 V_2(R) = (2/3)\pi^2 R^4$.

This is closer to the predicted quantity $(1/2)\pi^2 R^4$ but it is still too large by the value of the integral of the second expression above.

It would be possible to integrate the second expression by parts and recombine the expressions to get the final answer, but it is somewhat better to use another technique of integration, trigonometric substitution. This is a natural technique since we used the Pythagorean theorem to determine the integrand in the first place, and by looking at the diagram we used, we obtain all the quantities we need for a substitution: $x = R\sin(t), \sqrt{R^2 - x^2} = R\cos(t)$, $dx/dt = R\cos(x)$. As x goes from $-R$ to R, t goes from $-\pi/2$ to $\pi/2$. (Note that we could have chosen a substitution $x = \cos(t)$ based on a different triangle, but this leads to integrals that are more difficult to handle.)



If we make this trigonometric substitution in the integral for the area of the disc in the plane, we get $\int_{-R}^{R} 2\sqrt{R^2 - x^2}\,dx = \int_{-\pi/2}^{\pi/2} 2R\cos(t)(dx/dt)\,dt = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(t)\,dt$ Using the trigonometric identity $\cos(2t) = 2\cos^2(t) - 1$, we have $\int_{-\pi/2}^{\pi/2} 2R^2\cos^2(t)\,dt = \int_{-\pi/2}^{\pi/2} R^2(\cos(2t)+1)$. By symmetry of the cosine graph, the integral of the first expression is zero, so the area is just $\int_{-\pi/2}^{\pi/2} R^2 = \pi R^2$.

In 3-space, the integration is easier. The integral of $\pi(\sqrt{R^2 - x^2})^2$ with respect to x from $-R$ to R transforms to $\int_{-R}^{R} \pi(R^2 - x^2)\,dx = \int_{-\pi/2}^{\pi/2} \pi R^2\cos^2(t)(dx/dt)\,dt = \int_{-\pi/2}^{\pi/2} \pi R^3\cos^3(t)\,dt = \int_{-\pi/2}^{\pi/2} \pi R^3(1 - \sin^2(t))\cos(t)\,dt = \pi R^3(\sin(t) - \sin^3(t)/3)|_{-\pi/2}^{\pi/2} = 2(1 - 1/3)\pi R^3 = (4/3)\pi R^3 = V_3(R)$.

The analogy is now clear. To get the volume of the 4-ball of radius R, we integrate $(4/3)\pi(\sqrt{R^2 - x^2})^3$ with respect to t from $-R$ to R, and the trigonometric substitution transforms this to the integral of $\int_{-\pi/2}^{\pi/2}(4/3)\pi R^3\cos^3(t)(dx/dt)\,dt$. Thus the hypervolume is $\int_{-\pi/2}^{\pi/2}(4/3)\pi R^4\cos^4(t)(dx/dt)\,dt$

Now we use the trigonometric identity twice: $\cos^4(t) = (\cos^2(t))^2 = ((\cos(2t) + 1)/2)^2 = (\cos^2(2t) + 2\cos(2t) + 1)/4$. Since $\cos^2(2t) = (\cos(4t) + 1)/2$ we obtain the identity $\cos^4(t) = \cos(4t)/4 + \cos(2t)/2 + 3/8$. Once again by symmetry, the integral of the first two terms will be 0. Thus $\int_{-\pi/2}^{\pi/2}(4/3)\pi R^4\cos^4(t)(dx/dt)\,dt = (3/8)\pi$ and the desired hypervolume is $(4/3)\pi R^4(3/8)\pi = (1/2)\pi^2 R^4 = V_4(R)$, *quod erat demonstrandum*!

So, now what about higher dimensions? It should be very clear by now that the key is the computation of integrals of powers of the cosine. Let $C(n) = \int_{-\pi/2}^{\pi/2} \cos^n(t)(dx/dt)\,dt$, so, by our calculations above, $C(1) = 2, C(2) = (1/2)\pi, C(3) = 4/3$, and $C(4) = (3/8)\pi$. We might already conjecture at this point that for odd n we obtain a rational number and for even n, a rational number multiplied by π. This conjecture is reinforced when we calculate, as above, that $C(5) = 16/15$ and $C(6) = (5/16)\pi$.

We can use these values to find the formulas for the volumes of the balls of radius R in various dimensions: $V_1(R) = 2R$, $V_2(R) = \pi R^2$,$V_3(R) = (4/3)R^3$,$V_4(R)(1/2)\pi^2 R^4$. Note that $V_4(R) = R^4\pi^2(1/2) = R^4 V_3(1)C(4)$, and in general $V_n(R) = R^n V_{n-1}(1)C(n)$. Thus $V_5(R) = R^5 V_4(1)C(5) = R^5(\pi^2/2)(16/15) = R^5\pi^2(8/15)$ and $V_6(R) = R^6 V_5(1)C(6) = R^6\pi^2(8/15)(5/16)\pi = R^6\pi^3(1/2\cdot 3)$.

This last expression is very suggestive, coupled with the formulas for the volumes in the other even dimensions: $V_2(R) = \pi R^2, V_4(R) = \pi^2 R^4(1/2), V_6(R) = \pi^3 R^6(1/6)$. The conjecture for even dimensions $n = 2m$ seems quite clear: $V_{2m}(R) = \pi^m R^{2m}(1/m!)$. This turns out to be correct, and it can be established using mathematical induction. The corresponding formula for odd dimensions $n = 2m + 1$ is slightly more complicated: $V_{2m+1}(R) = 2^{2m+1}\pi^m R^{2m+1} m!/(2m+1)!$, and this can also be extablished by mathematical induction. In each case, a key step is the *recursion formula* $C(n) = [(n-1)/n]C(n-2)$, which follows immedaitely by integration by parts.

These final formulas are very close to the ones we were developing in the last hour of our session. Enjoy mathematics, and keep on asking "What about higher dimensions?"