# On the Number of Extrema of a Polyhedron †

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**Abstract:** In this paper we obtain a tight bound on the number of local minima and maxima (collectively called *extrema*) of a three-dimensional polyhedron $P$ with respect to an arbitrary direction $\lambda$. For simplicity, we assume that no edge of $P$ is normal to $\lambda$, which implies that only vertices of $P$ can be extrema; if we relax this assumption, our bound translates into a bound on the number of maximal connected sets of extrema. In two dimensions, an easy inductive proof shows that the number of extrema of a polygon (possibly with holes) that has $r$ reflex vertices is at most $r + 2$; this bound proved instrumental in the analysis of the polygon triangulation algorithm of Hertel and Mehlhorn [7]. We are interested in the three dimensional counterpart. We prove that if the polyhedron has $r$ reflex angles, then the number of extrema (with respect to a given direction) does not exceed $2r + 2$. Moreover, for every integer $r \geq 0$, we exhibit a polyhedron that has $r$ reflex angles and $2r + 2$ extrema with respect to a certain direction.

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## 1. Introduction.

Consider a three-dimensional polyhedron $P$ and an arbitrary oriented line $\lambda$; we are interested in computing bounds on the number of *extrema* of $P$ (that is, its local minima and maxima) with respect to $\lambda$. For simplicity, we assume that no edge of $P$ is normal to $\lambda$; not only does this assumption imply that only vertices of $P$ can be extrema, and thus guarantees that there is indeed an upper bound, but such a bound translates into a bound on the number of *maximal connected sets* of extrema if the assumption is relaxed (note that in the latter case, an entire edge or an entire face, and thus an infinite number of points, may contribute extrema with respect to $\lambda$).

Except for its importance as a combinatorial result, a bound on the number of extrema proves to be useful in estimating the complexity of algorithms. For example, in two dimensions, a $2r + 2$ upper bound on the number of extrema of a polygon that has $r$ reflex vertices proves instrumental in the analysis of Hertel and Mehlhorn's algorithm for polygon triangulation [7]. In turn, our interest in the three-dimensional version of the problem is motivated from an algorithm to decompose the boundary of a polyhedron into "well behaved" patches [4], where the number of produced patches relies on the number of extrema of the polyhedron. It is important to observe that the number of extrema of a polygon or polyhedron is directly related to how "non-convex" the polygon or polyhedron is; after all, a convex object exhibits exactly two extrema. So, it is desirable that the bound on the number of extrema be expressed in terms of a measure of non-convexity of the polygon or polyhedron. For a polygon, it is the number of its reflex vertices that is most commonly used for this purpose; for a polyhedron it is the number of its reflex edges, i.e., edges whose corresponding (internal) dihedral angles exceed $\pi$. Several recent algorithms and bounds have been expressed in terms of these measures (see for example [2], [6], and [1], [3], [5]).

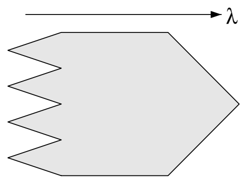

Figure 1

The problem of computing bounds on the number of extrema of a polygon (that may have holes) is very simple. If $r$ denotes the number of reflex vertices of the polygon, we can prove by induction that the number of extrema with respect to any given direction does not exceed $r + 2$: the inductive step involves "resolving" a reflex vertex $p$ by introducing a cut that forms acute angles with $p$'s incident edges; note that the cut may split the polygon. The bound is tight, since for every integer $r \geq 0$ there exists a polygon that has $r$ reflex vertices and

$r+2$ extrema (see Figure 1). This approach does not extend to the three-dimensional case, however; "resolving" a reflex edge may lead to splitting other reflex edges, thus effectively doubling their number in the worst case. Our approach relies on observing how the number of polygons changes in the intersection of a polyhedron $P$ with a sweep plane that moves along a direction $\lambda$; it helps us prove an upper bound of $2r+2$ on the number of extrema of $P$ with respect to $\lambda$, where $r$ is the number of reflex edges of $P$. We also show that this bound is tight by exhibiting a polyhedron that has $r$ reflex edges and $2r+2$ extrema (for every integer $r \geq 0$).

The paper is structured as follows. In Section 2 we introduce our terminology. The upper bound is proved in Section 3, while the lower bound is presented in Section 4. Finally, in Section 5 we summarize our results and pose some open questions.

## 2. Geometric Framework.

A polyhedron in $\mathcal{R}^3$ is a connected piecewise-linear 3-manifold with boundary. Its boundary is connected and it consists of a collection of relatively open sets, the *faces* of the polyhedron, which are called *vertices*, *edges*, or *facets*, if their affine closures have dimension 0, 1, or 2, respectively. The definition of a polyhedron rules out self-intersecting, dangling, or abutting faces, as well as degeneracies like the ones shown in Figure 2 (the shown objects have locally been compressed into a single point or a single edge). Then, each edge is incident upon exactly two facets; if the (interior) dihedral angle formed by the facets incident upon an edge $e$ of a polyhedron exceeds $\pi$, we say that $e$ is *reflex*.

Next, we formally define the notion of extrema in any dimension $d$. A point $p$ of a $d$-dimensional set $S$ of points is called an *extremum* of $S$ with respect to an oriented line $\lambda$, or a $\lambda$-*extremum* for short, if the intersection of $S$ with a small enough $d$-ball centered at $p$ lies entirely in the one of the two closed halfspaces defined by the hyperplane normal

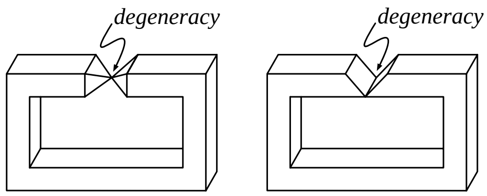


Figure 2

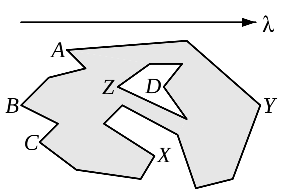


Figure 3

to $\lambda$ that passes through $p$. The extrema are characterized as *negative* or *positive* depending on whether the above intersection lies in the non-negative or non-positive halfspace respectively. For the polygon of Figure 3, for instance, the vertices $A$, $B$, $C$ and $D$ are negative $\lambda$-extrema, while $X$ and $Y$ are positive ones. The vertex $Z$ is *not* an extremum. The definition allows for all the points of an edge or even a facet of a polyhedron $P$ to be extrema, which obviously defeats the purpose of computing an upper bound on their number. If we assume, however, that no edge of $P$ is normal to the line $\lambda$, then only vertices of $P$ can be extrema, and hence their number is bounded; moreover, a bound on the number of extrema under the above assumption translates into a bound on the number of maximal connected sets of extrema if the assumption is relaxed.

## 3. The Upper Bound.

Without loss of generality, we assume that we wish to compute the number of extrema with respect to the $x$-axis of a polyhedron $P$. We additionally assume that no edge of $P$ is normal to the $x$-axis, so that we have a unique point-representative for every connected set of extrema; otherwise, we can use a lexicographic ordering, which essentially corresponds to an infinitesimal tilting of the polyhedron. The basic idea is to sweep $P$ with a plane normal to the $x$-axis, and observe the changes in the intersection of $P$ with the plane: we call such an intersection a *$yz$-cross-section* of $P$. It is important to observe that for as long as the plane sweeps the polyhedron without encountering a vertex, the corresponding $yz$-cross-section changes conformally; therefore, we need to concentrate on how the $yz$-cross-section changes, as the plane passes through a vertex; in particular, we need to focus only on the neighborhood of that vertex in the cross-section. For simplicity, we use $\mathcal{C}_{yz}(v)$ to denote the $yz$-cross-section when the sweep plane is located at the vertex $v$ of the polyhedron; similarly, $\mathcal{C}^-_{yz}(v)$ $\big(\mathcal{C}^+_{yz}(v)$ resp.$\big)$ denotes the cross-sections infinitesimally before (after resp.) the vertex $v$ is reached. Depending on the geometry of the neighborhood of $v$ in $\mathcal{C}_{yz}(v)$, we distinguish the following cases:

1. **The neighborhood of $v$ in $\mathcal{C}_{yz}(v)$ is a point-polygon** (Figure 4): Then, $v$ is an $x$-extremum. Figure 4 depicts how the cross-section changes from $\mathcal{C}^-_{yz}(v)$ to $\mathcal{C}^+_{yz}(v)$, if $v$ is a negative $x$-extremum; the case for a positive $x$-extremum is obtained by interchanging the figures for $\mathcal{C}^-_{yz}(v)$ and $\mathcal{C}^+_{yz}(v)$. The definition of the

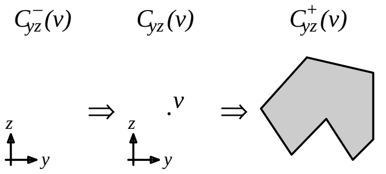

Figure 4

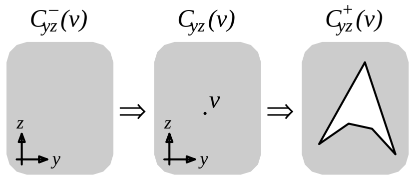

Figure 5

polyhedron implies that as the sweep plane passes through $v$, either *exactly one* new polygon appears, or *exactly one* polygon disappears in the cross-section.

2. **The neighborhood of $v$ in $\mathcal{C}_{yz}(v)$ is a point-hole in a polygon** (Figure 5): Then, $v$ is not an $x$-extremum, but it is like the vertex $Z$ in Figure 3. Figure 5 depicts one of the two basic cases that may arise; the other one results from Figure 5 after the figures for $\mathcal{C}^+_{yz}(v)$ and $\mathcal{C}^-_{yz}(v)$ have been interchanged. In this case, the number of polygons in the cross-section does not change; instead, either one new hole in a polygon appears, or one such hole disappears.

3. **The degree of $v$ in $\mathcal{C}_{yz}(v)$ is 2** (Figure 6): In this case, $v$ belongs to a single polygon, which may exhibit corrugations in the neighborhood of $v$ in $\mathcal{C}^+_{yz}(v)$ (Figure 6 shows two examples). In any case, the total number of polygons in the cross-section does not change.

4. **The vertex $v$ is of degree larger than 2 in $\mathcal{C}_{yz}(v)$:** This is the really interesting case; this time, the neighborhood of $v$ in $\mathcal{C}_{yz}(v)$ consists of, say, $w_v$ wedges ($w_v \geq 2$) that touch at $v$ ($w_v = 6$ in Figure 7); the dashed curves indicate that some of the wedges may belong to the same polygon of $\mathcal{C}_{yz}(v)$. These wedges may either merge with or split from other wedges in $\mathcal{C}^+_{yz}(v)$ and $\mathcal{C}^-_{yz}(v)$. In Figure 7 for instance, as $\mathcal{C}_{yz}(v)$ evolves into $\mathcal{C}^+_{yz}(v)$, wedges 2, 4, and 6 merge into a single wedge-cluster, while each of the remaining three wedges forms a wedge-cluster by itself. Let $w^+_v$ and $k^+_v$ ($w^-_v$ and $k^-_v$) denote the number of such wedge-clusters and the number of polygons containing these clusters in $\mathcal{C}^+_{yz}(v)$ ($\mathcal{C}^-_{yz}(v)$) respectively

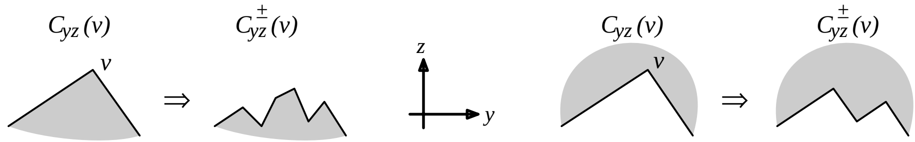

Figure 6

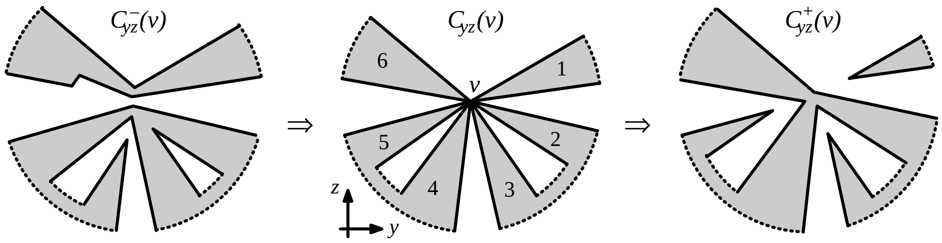


Figure 7

(in Figure 7, we have $w_v^- = 3$, $w_v^+ = 4$, and $k_v^- = k_v^+ = 2$). Since wedges may merge in $\mathcal{C}^-_{yz}(v)$ or $\mathcal{C}^+_{yz}(v)$, and a polygon may contain more than one such wedge,

$$w_v \;\geq\; w_v^+ \;\geq\; k_v^+ \qquad \text{and} \qquad w_v \;\geq\; w_v^- \;\geq\; k_v^-. \tag{1}$$

Moreover, it is crucial to observe that:

**Observation 1**:
$$w_v^- + w_v^+ \;\leq\; w_v + 1. \tag{2}$$

This is a consequence of the definition of a polyhedron given in Section 2. To prove the observation, we construct a graph on $w_v$ points each corresponding to one of the $w_v$ wedges: for each wedge-cluster in either $C^-_{yz}$ or $C^+_{yz}$, we draw a tree spanning the points corresponding to the wedges that merge to form the wedge-cluster; an arbitrary tree will do, since the goal is to record the connectivity information. (Figure 8 shows the corresponding graph for the yz-cross-sections of Figure 7; the edges corresponding to wedge-clusters in $C^-_{yz}$ are drawn solid, those of wedge-clusters in $C^+_{yz}$ are drawn dashed.) Therefore, for a wedge-cluster that involves $k$ wedges, $k-1$ edges are introduced. Then, all the wedge-clusters in $C^-_{yz}$ contribute $w_v - w_v^-$ edges, since each wedge belongs to exactly one wedge-cluster in $C^-_{yz}$ and there are $w_v^-$ such clusters. Similarly, the wedge-clusters in $C^+_{yz}$ contribute a total of $w_v - w_v^+$ edges. Now, according to the polyhedron's definition, the neighborhood of $v$ is homeomorphic to the positive 3d-octant, which implies that the above graph is connected; hence, the total number of tree edges is at least equal to $w_v - 1$. In other words,

$$(w_v - w_v^-) + (w_v - w_v^+) \quad\geq\quad w_v - 1,$$

and inequality (2) follows.

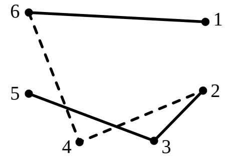


Figure 8

Another important observation is:

**Observation 2**: if a vertex $v$ is incident upon $w_v$ wedges in $\mathcal{C}_{yz}(v)$, then it is incident upon *at least* $w_v - 1$ reflex edges of $P$.

This follows from the fact that when $k$ wedges merge, they exhibit at least $k-1$ reflex edges at the merging "seams" (see Figure 7).

Depending on the case in which it falls, each vertex of $P$ is put into one of four sets, $V_1$, $V_2$, $V_3$, and $V_4$ corresponding to cases 1, 2, 3, and 4, respectively. Then,

**Lemma 3.1.** *The total number $|V_1|$ of positive and negative $x$-extrema of a polyhedron does not exceed $\sum_{v \in V_4} (w_v - 1) + 2$, where $w_v$ is as defined earlier.*

*Proof:* Again, we assume that we sweep the polyhedron $P$ with a plane normal to the $x$-axis. We construct a graph that records the number of polygons in the history of the $yz$-cross-section; we concentrate only on the cases where the plane encounters positive or negative $x$-extrema, or vertices in $V_4$, since it is precisely then when the number of polygons changes. In particular,

1. at a negative $x$-extremum (vertex in $V_1$), we add two new nodes[‡] in the graph which we connect by an edge; the first node corresponds to the negative $x$-extremum while the second one is a *polygon-node* and corresponds to the series of polygons into which the negative $x$-extremum evolves;
2. at a positive $x$-extremum (vertex in $V_1$), we add one new node that corresponds to the positive $x$-extremum and we connect it to the polygon-node that represents the polygon which shrunk to this positive $x$-extremum during the sweeping;
3. at a vertex $v$ in $V_4$, we add one new node that corresponds to $v$ and edges connecting it to the representatives of the $k_v^-$ polygons in $\mathcal{C}_{yz}^-(v)$. Moreover, $k_v^+$ polygon-nodes are added, one for each of the $k_v^+$ polygons in $\mathcal{C}_{yz}^+(v)$, and edges are introduced between them and the node corresponding to $v$.

Since we are dealing with a single polyhedron, the resulting graph is connected. Figure 9(b) shows the graph that corresponds to the polyhedron of Figure 9(a). The numbers 1, 2, and 3 denote negative and positive $x$-extrema, and vertices in $V_4$ respectively (in agreement with the cases above), while the letter $P$ denotes polygon-nodes.

[‡] We refer to *nodes* of a graph (instead of vertices) to avoid confusion with the vertices of the polyhedron.

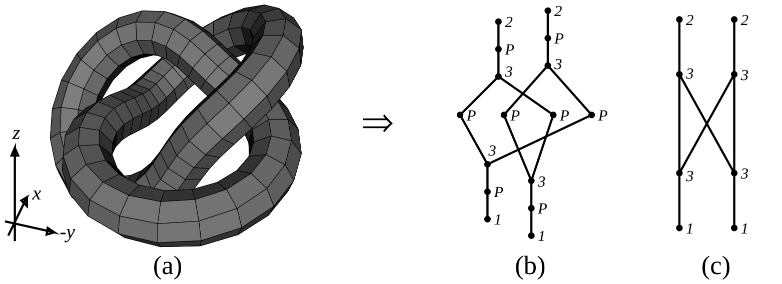


Figure 9

Note that all polygon-nodes are of degree 2, and no two of them are adjacent. Moreover, the nodes corresponding to negative and positive $x$-extrema of the polyhedron are of degree 1.

To simplify matters, we remove from the graph all polygon-nodes by coalescing their incident edges into a single edge; the resulting graph is a connected multigraph (see Figure 9(c)), whose node set is in one-to-one correspondence with the union of $V_1$ and $V_4$. If we denote by $m$ the total number of edges of the multigraph, we have

$$2\,m \;=\; \sum_{\text{node } v} \text{degree}(v) \;=\; \sum_{v\in V_1} 1 \;+\; \sum_{v\in V_4} (k_v^- + k_v^+) \;\le\; |V_1| + \sum_{v\in V_4} (w_v + 1), \qquad (3)$$

since, for any vertex $v \in V_4$, inequalities (1) and (2) imply that $k_v^- + k_v^+ \le w_v^- + w_v^+ \le w_v + 1$. Connectivity, on the other hand, implies that the number of edges is at least equal to 1 less than the number of nodes of the graph, that is, $m \ge |V_1| + |V_4| - 1$, which combined with (3) yields

$$2\Big(|V_1| + |V_4| - 1\Big) \;\le\; |V_1| + \sum_{v\in V_4} (w_v + 1).$$

The lemma follows. ∎

The desired upper bound on the number of extrema of a polyhedron follows as a consequence of Lemma 3.1 and Observation 2.

**Theorem 3.1.** *The total number of positive and negative $x$-extrema of a polyhedron with $r$ reflex edges does not exceed $2r + 2$.*

*Proof:* According to Observation 2, the sum $\sum_{v\in V_4} (w_v - 1)$ is at least equal to twice the number of reflex edges incident upon the vertices in $V_4$, which in turn is at least equal to $2r$. ∎

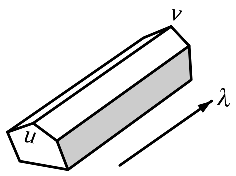


Figure 10

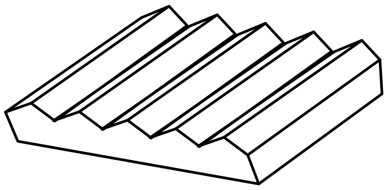

Figure 11

## 4. The Lower Bound.

The above upper bound is in fact tight, since there is a polyhedron with $r$ reflex edges that has $2r + 2$ extrema (for arbitrary $r$). Our construction involves the basic polyhedron shown in Figure 10: it looks like a house whose corners $u$ and $v$ have been pulled outwards along the direction $\lambda$; $u$ and $v$ are the extrema with respect to $\lambda$. By gluing $r + 1$ of these polyhedra together at their shaded facets, we get a polyhedron with $r$ reflex edges and $2(r + 1)$ extrema with respect to $\lambda$ (Figure 11).

## 5. Conclusions.

In this paper, we established a tight bound on the number of extrema of a three-dimensional polyhedron in terms of the number $r$ of its reflex edges. We first proved a $2r + 2$ upper bound, which we matched by exhibiting a polyhedron that has $r$ reflex edges and $2r + 2$ extrema with respect to some direction, for every integer $r \geq 0$.

The open question that immediately comes to mind is what happens in higher dimensions: what is the number of extrema (with respect to some direction) of a $d$-dimensional polyhedron? What is needed first, however, is a measure of non-convexity of such a polyhedron. How do the notions of reflex vertices of a polygon and reflex edges of a 3-dimensional polyhedron extend in four and higher dimensions?

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