Symmetries, Isometries and Length Spectra of Closed Hyperbolic 3-Manifolds

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ABSTRACT

Previously known algorithms for computing the symmetry group of a cusped hyperbolic 3-manifold and testing whether two cusped hyperbolic 3-manifolds are isometric do not apply directly to closed manifolds. But by drilling out geodesics from closed manifolds we may compute their symmetry groups and test for isometries using the cusped manifold techniques. To do so, we must know precisely how many geodesics of a given length the closed manifold has. This article proves the propositions needed to rigorously compute a length spectrum, with multiplicities, and goes on to tabulate the symmetry groups of the smallest known closed hyperbolic 3-manifolds.

Keywords: hyperbolic 3-manifold, length spectrum, symmetry, isometry.

0. Introduction. A canonical cell decomposition underlies algorithms for deciding whether two cusped hyperbolic 3-manifolds are isometric [HiW], and for computing the symmetry group of a cusped hyperbolic 3-manifold [HeW]. The canonical cell decomposition depends on a convex hull construction in Minkowski space [EP, W, SW]. That construction makes essential use of a manifold's cusps, and does not generalize directly to closed manifolds. Fortunately we can transfer questions about closed manifolds to questions about cusped manifolds by looking at the complement of a set of nonintersecting closed geodesics.

Notation. Throughout this article, M denotes a finite-volume hyperbolic 3-manifold. In this Introduction we further assumed that M is closed. We do not assume M is orientable, because our results, as well as the results we quote from [K] and [S], apply equally well to nonorientable manifolds.

Definition. A *geodesic link* L is a finite set of closed geodesics in M.

Definition. A geodesic link L is *simple* iff its constituent geodesics do not intersect themselves or each other.

Proposition 0.1 (Sakai [S])**.** If L is a simple geodesic link in M, then its complement M - L admits a complete hyperbolic structure of finite volume.

Definition. A geodesic link L is *complete* iff whenever it contains one geodesic of a given length, it contains all geodesics of that length. That is, for any two closed geodesics γ and γ' in M, if $\gamma \in L$ and length(γ) = length(γ'), then $\gamma' \in L$.

Proposition 0.2 (Kojima [K])**.** If L is a complete simple geodesic link in a closed hyperbolic 3-manifold M, then Isom(M) is naturally isomorphic to the subgroup of Isom(M - L) consisting of symmetries which take meridians to meridians (i.e. symmetries which extend to M).

Proof. Follows immediately from Lemma 5 and Proposition 6 of [K]. The basic idea is that a symmetry of M must preserve the geodesic link L, and therefore restricts to a homeomorphism of M - L to itself. By Mostow rigidity, the homeomorphism is homotopic to an isometry. *Q.E.D.*

Proposition 0.2 lies at the heart of our method for computing symmetry groups of closed manifolds. Given a closed manifold M, we drill out a complete simple geodesic link L, compute the symmetry group Isom(M - L) of its complement using the canonical cell decomposition mentioned above, and then take the subgroup of Isom(M - L) which sends meridians to meridians. Section 1 addresses the question of how we find a complete simple geodesic link, and Section 2 tabulates the symmetry groups of the ten smallest known hyperbolic 3-manifolds.

A similar technique lets us test whether two closed hyperbolic 3-manifolds M and M' are isometric: we drill all the geodesics of a given length out of both M and M', and then use canonical decompositions to check whether $M - L$ and $M' - L'$ are isometric by a meridian-preserving isometry.

The computer program SnapPea implements the above algorithms, as well as other algorithms for creating and studying hyperbolic 3-manifolds. It is available free of charge by anonymous ftp from geom.umn.edu, or on a Macintosh diskette from the second-named author.

1. Length spectra. For over a year now the computer program SnapPea has had the ability to drill closed geodesics from hyperbolic 3-manifolds. One could use it informally to compute a closed manifold's symmetry group. The problem was that the results weren't rigorous. The drilling program sees only the geodesics which lie in the dual 1-skeleton of the given triangulation: one can usually find any geodesic one wants to drill out (perhaps after changing the triangulation), but from the standpoint of rigor this is unsatisfactory because one is never entirely sure of what geodesics may be missing. If the geodesic link L (defined above) is not complete, then the computed symmetry group will be a subgroup of the true symmetry group, namely the subgroup which fixes the (incomplete) link L.

To make the technique rigorous, we need to know how many geodesics of a given length appear in M. In other words, we need a length spectrum with multiplicities. The remainder of this section explains SnapPea's new algorithm for rigorously computing such a length spectrum.

The key idea is to begin with a Dirichlet domain for M.

Notation. Throughout this section M may be any finite-volume hyperbolic 3-manifold or 3-orbifold.

Notation. Let D be a Dirichlet domain for M with basepoint x. A typical translate of D will be denoted gD, where g is an isometry in the group of covering transformations. If M is an orbifold, the basepoint x must not lie in the singular set.

We'll tile an approximate ball in $\mathbf{H}^3$ with translates gD of D. Each primitive hyperbolic or loxodromic isometry g corresponds to a translation along a geodesic, perhaps followed by a rotation or reflection which fixes the geodesic pointwise. If g is orientation-preserving and is given as a matrix in $PSL(2,\mathbf{C})$, then the trace of the matrix determines the complex length of the geodesic (cf. [B], §4.3). If g is orientation-reversing, then a similar technique applies, only now the Minkowski space model of $\mathbf{H}^3$ is more convenient and we use matrices in $SO(3,1)$. Let's take a quick look at the mathematics of orientation-reversing isometries before continuing with the discussion of length spectra.

Proposition 1.1. Each orientation-reversing isometry of $\mathbf{H}^3$ is a reflection in a plane, followed by an orientation-preserving isometry which fixes that plane (setwise, not pointwise!).

Sketch of proof. Classify the orientation-reversing isometries of $\mathbf{H}^3$ according to whether they do or do not have a fixed point in the interior of $\mathbf{H}^3$, and then for the ones which do not, classify them further according to whether they have 1, 2 or infinitely many fixed points on the sphere at infinity. *Q.E.D.*

Definition. An orientation-reversing isometry of $\mathbf{H}^3$ is *elliptic*, *parabolic* or *hyperbolic* iff it factors as a reflection in a plane, followed by an orientation-preserving elliptic, parabolic or hyperbolic isometry, respectively, which fixes that plane (setwise).

Comments.
(1) Orientation-reversing hyperbolic isometries are glide reflections.
(2) Orientation-reversing loxodromics do not exist.
(3) A simple reflection in a plane corresponds to the identity in the orientation-preserving case. So it may be considered a degenerate elliptic, parabolic or hyperbolic isometry.

Proposition 1.2. Orientation-reversing isometries have traces in $SO(3,1)$ as follows:

elliptic	$2\cosh(i\theta) = 2\cos(\theta)$
parabolic	2
hyperbolic	$2\cosh(\mathit{length})$

Sketch of proof. Follows from the classification of orientation-reversing isometries alluded to in the proof of Proposition 1.1. The trace is a conjugacy invariant, so we compute it in the simplest coordinate system we can find for each type of isometry. *Q.E.D.*

Corollary 1.3. An orientation-reversing isometry of $\mathbf{H}^3$ is elliptic, parabolic, or hyperbolic according to whether the trace of its matrix in $SO(3,1)$ is less than, equal to, or greater than 2. If it's elliptic, the rotation angle is $\cos^{-1}(\mathit{trace}/2)$. If it's hyperbolic, the glide reflection translates a distance $\cosh^{-1}(\mathit{trace}/2)$ along its axis.

Proof. Follows immediately from Proposition 1.2. *Q.E.D.*

We now return to the discussion of length spectra. We'd like to find all translates gD of D which move the basepoint x a distance less than some given distance s, i.e. we'd like to find all gD such that $d(x, gx) < s$. The simplest algorithm is to start with D and recursively attach its neighbors, stopping the recursion when we reach translates gD which move the basepoint a distance greater than s, i.e. when $d(x, gx) > s$. For an arbitrary fundamental domain (not necessarily a Dirichlet domain) with an arbitrary basepoint, this algorithm might fail: there could be a translate moving the basepoint a distance less than s, all of whose neighbors move the basepoint a distance greater than s. The simple recursive algorithm would not find such a translate. Fortunately this cannot occur for a Dirichlet domain.

Proposition 1.4. Let D be a Dirichlet domain with basepoint x, and let gD be a translate of D such that for all neighbors hD of gD, $d(x, hx) \geq d(x, gx)$. Then g is the identity.

Proof. For each neighbor hD of gD, the inequality $d(x, hx) \geq d(x, gx)$ implies that x lies in the halfspace H_h consisting of points closer to gx than hx. But gD is the intersection of all such H_h, so x must lie in gD. Therefore $gD = D$. *Q.E.D.*

If we want to find all geodesics of length at most λ, to what radius s must we tile $\mathbf{H}^3$? The answer to this question depends not only on λ, but also on the size of the Dirichlet domain. The relevant measure of the Dirichlet domain's size is its "spine radius", defined below. Proposition 1.5 provides a practical means of computing the spine radius. Proposition 1.6 then expresses the required tiling radius s in terms of the spine radius r and cutoff length λ.

The Dirichlet domain D, with faces identified, defines a cell decomposition K for M. Let K' be a cell decomposition dual to K. The 3-cells of K' are in one-to-one correspondence with the vertices of K. If M is a manifold, each 3-cell of K' will be either a topological ball (if the corresponding vertex of K is finite) or a topological torus or Klein bottle cross a half line (if the corresponding vertex of K is ideal). If M is an orbifold, a 3-cell dual to a finite vertex may be a cone on any spherical 2-orbifold, and a 3-cell dual to an infinite vertex may be any Euclidean 2-orbifold cross a half line. We know the 3-cells must take this form, because the singular set (if any) must be contained in the Dirichlet domain's boundary (this follows from our assumption that the basepoint x not lie in the singular set). The essential point is that no geodesic may lie entirely in a single 3-cell of K'. It must intersect the 2-skeleton of K'.

Definition. Let K' be a dual cell decomposition as defined in the preceding paragraph. Its 2-skeleton is a *spine dual to the Dirichlet domain*.

Definition. The *radius* of a spine dual to the Dirichlet domain is the maximum distance from a point in the spine to the basepoint.

Definition. A Dirichlet domain's *spine radius* is the infimum of the radii of all spines dual to the Dirichlet domain.

Definition (local to Proposition 1.5). The *"maximin" edge distance* is the maximum over all the Dirichlet domain's edges of the minimum distance from the edge to the basepoint x.

Proposition 1.5. The spine radius may be computed as the maximin edge distance.

Proof. Any spine dual to the Dirichlet domain must intersect every edge, so the spine radius is greater than or equal to the maximin edge distance. It remains to show that for any ε greater than zero, we can construct a spine whose radius is within ε of the maximin edge distance. Carry out the following construction in the Dirichlet domain D (not the space M obtained by identifying faces).

Step 1. On each edge, mark the point closest to the basepoint. If that point is at an endpoint, displace it a distance ε into the interior of the edge. Note that
- (a) the edge identifications respect the marked points, and
- (b) the marked points all lie within the maximin edge distance plus ε of the basepoint.

Step 2. On each face, mark the point closest to the basepoint. If that point is on the boundary, displace it a distance ε into the interior of the face (towards the center, say). Note that
- (a) the face identifications respect the marked points, and
- (b) the marked points all lie within the maximin edge distance plus ε of the basepoint.

Step 3. Draw lines from the marked point in the interior of each face to the marked points on the incident edges. Note that
- (a) the face identifications respect the lines, and
- (b) the lines all lie within the maximin edge distance plus ε of the basepoint.

Step 4. Cone the complex created in steps (1), (2) and (3) to the basepoint. This gives a spine which is dual to the Dirichlet domain and lies within the maximin edge distance plus ε of the basepoint.

Q.E.D.

Comment. The spine radius is finite even for Dirichlet domains of cusped manifolds, whose radius is infinite. The techniques of this section work to compute length spectra for cusped manifolds as well as closed ones.

Performance enhancement. In actual practice SnapPea removes certain 2-cells from the spine to reduce its radius without compromising the fact that every geodesic must intersect it, but this modification does not affect the basic theory.

Notation. For the remainder of this section, let r be the spine radius of the Dirichlet domain D.

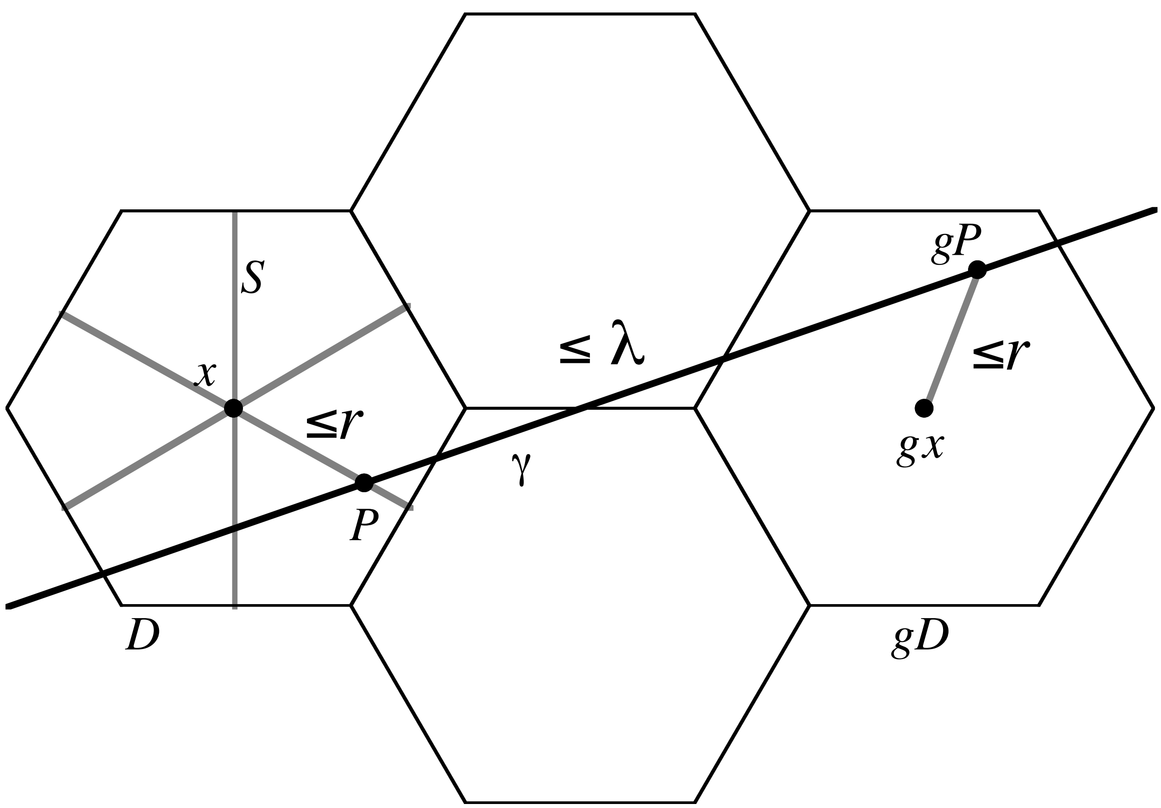


Figure 1.1. The distance from x to gx is at most $r + \lambda + r$.

Proposition 1.6.1. To find all closed geodesics of length at most λ, it suffices to find all translates gD satisfying $\mathrm{d}(x, gx) \leq \lambda + 2r$.

Proof. For each closed geodesic γ of length at most λ, we must find a covering transformation g whose axis is a preimage of the geodesic γ and which satisfies $\mathrm{d}(x, gx) < \lambda + 2r$.

Choose $\varepsilon > 0$, and construct a spine S of D with radius at most $r + \varepsilon$. (See Figure 1.1.) The geodesic γ must intersect the spine S at some point P, since otherwise γ would be contractible (or perhaps boundary parallel if M has cusps). Let g be the covering transformation corresponding to a lift of γ based at P. Then $\mathrm{d}(P, gP) = \mathrm{length}(\gamma)$, and $\mathrm{d}(x, gx) \leq \mathrm{d}(x, P) + \mathrm{d}(P, gP) + \mathrm{d}(gP, gx) \leq (r + \varepsilon) + \lambda + (r + \varepsilon) = \lambda + 2r + 2\varepsilon$. Let ε go to zero to obtain $\mathrm{d}(x, gx) \leq \lambda + 2r$. *Q.E.D.*

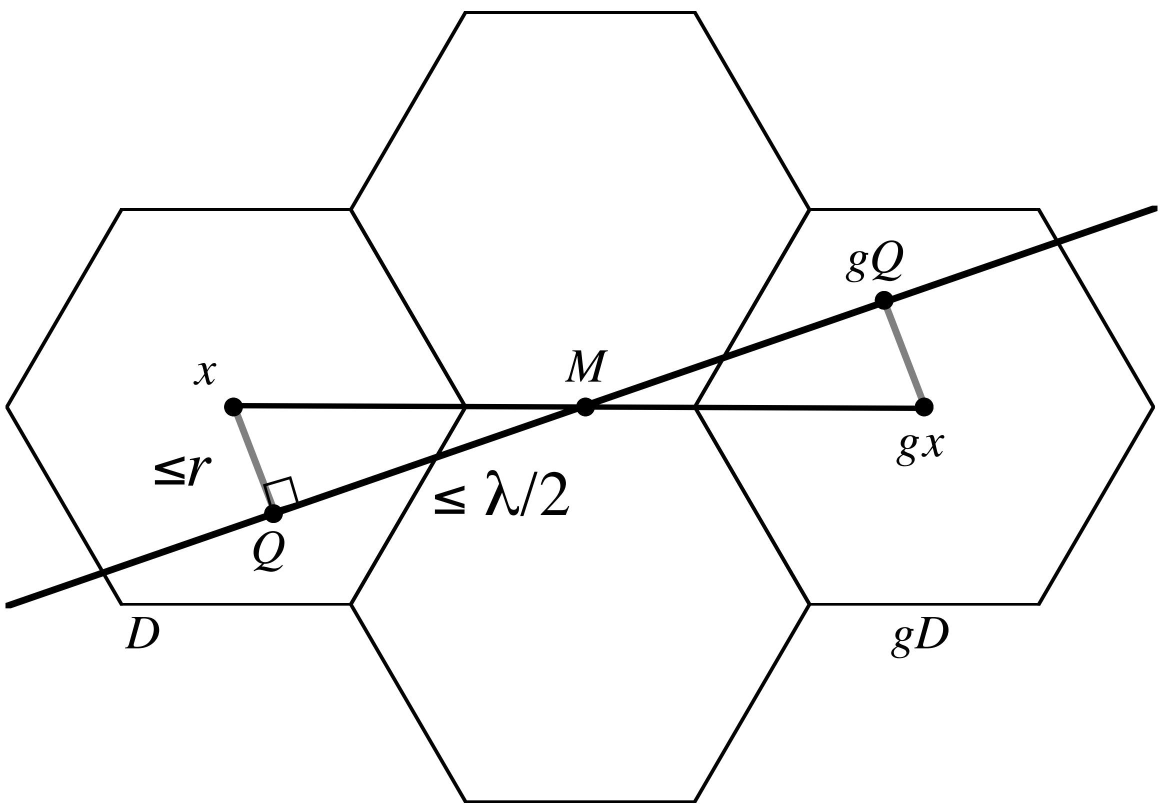


Figure 1.2. The distance from x to gx is at most $2 \cosh^{-1}(\cosh(r) \cosh(\lambda/2))$.

With a little extra care, we can improve on the estimate of $\lambda + 2r$.

Proposition 1.6.2. To find all closed geodesics of length at most λ, it suffices to find all translates gD satisfying $d(x, gx) \le 2 \cosh^{-1}(\cosh(r) \cosh(\lambda/2))$.

Proof. The proof is the same as that of Proposition 1.6.1, except instead of using a point P where γ intersects the spine, we use a point Q at which the distance from γ to the basepoint x is a minimum. (See Figure 1.2.) For all choices of ε and S we have $d(x, Q) \le d(x, P) \le r + \varepsilon$, so $d(x, Q) \le r$. The advantage of using Q instead of P is that the segment from x to Q is orthogonal to the geodesic γ. If we let M be the midpoint of the segment from Q to gQ we can use hyperbolic trigonometry to bound the distance from x to M as $d(x, M) \le \cosh^{-1}(\cosh(r) \cosh(\lambda/2))$. The distance from x to gx will be at most twice that. *Q.E.D.*

Comment. Empirical tests show that for small manifolds the length spectrum computation runs about five times faster with the bound from Proposition 1.6.2 than with the bound from Proposition 1.6.1. Theoretical estimates predict that as r and λ go to infinity, the run-time difference approaches a factor of 16.

Proposition 1.6.2 tells us how to find group elements corresponding to all geodesics of length at most λ. But different groups elements may be conjugate to one another and correspond to the same geodesic. To find a length spectrum with correct multiplicities, we must find precisely one group element in each conjugacy class.

Definition (local to the following algorithm).

(1) The *big list* is the complete set of group elements g satisfying $d(x, gx) \le 2 \cosh^{-1}(\cosh(r) \cosh(\lambda/2))$.

(2) The *small list* is a subset of the big list. Initially it equals the big list, but as the algorithm progresses we remove elements until precisely one element of each conjugacy class remains.

Algorithm. A four-part algorithm computes the small list from the big list:

(1) Remove group elements whose corresponding "geodesics" have length zero. If M is a manifold these will all be parabolics, but if M is an orbifold elliptics may also occur. Also remove group elements of length greater than λ.

(2) Remove group elements whose axes don't pass within a distance r of the basepoint. (Every geodesic must intersect a spine of radius $r + \varepsilon$, for all $\varepsilon > 0$, so we are sure to retain at least one element in every conjugacy class.)

(3) Remove group elements which are powers of other groups elements. We aren't interested in a "geodesic" which is just a shorter geodesic traversed twice.

(4) Check which of the elements remaining on the small list are conjugate to one another. Proposition 1.7 insures that if two elements of the small list are conjugate to each other, the conjugacy is realized by an element from the big list. That is, if two elements g_1 and g_2 on the small list are conjugate, there exists an element h on the big list such that $g_1 = h g_2 h^{-1}$. Keep precisely one element corresponding to each geodesic, and discard its conjugates, its inverse, and the conjugates of its inverse. (In a manifold an element and its inverse will be in different conjugacy classes, but in an orbifold they may be in the same conjugacy class, in which case the geodesic is topologically a mirrored interval rather than a circle.)

The small list now yields a length spectrum with correct multiplicities.

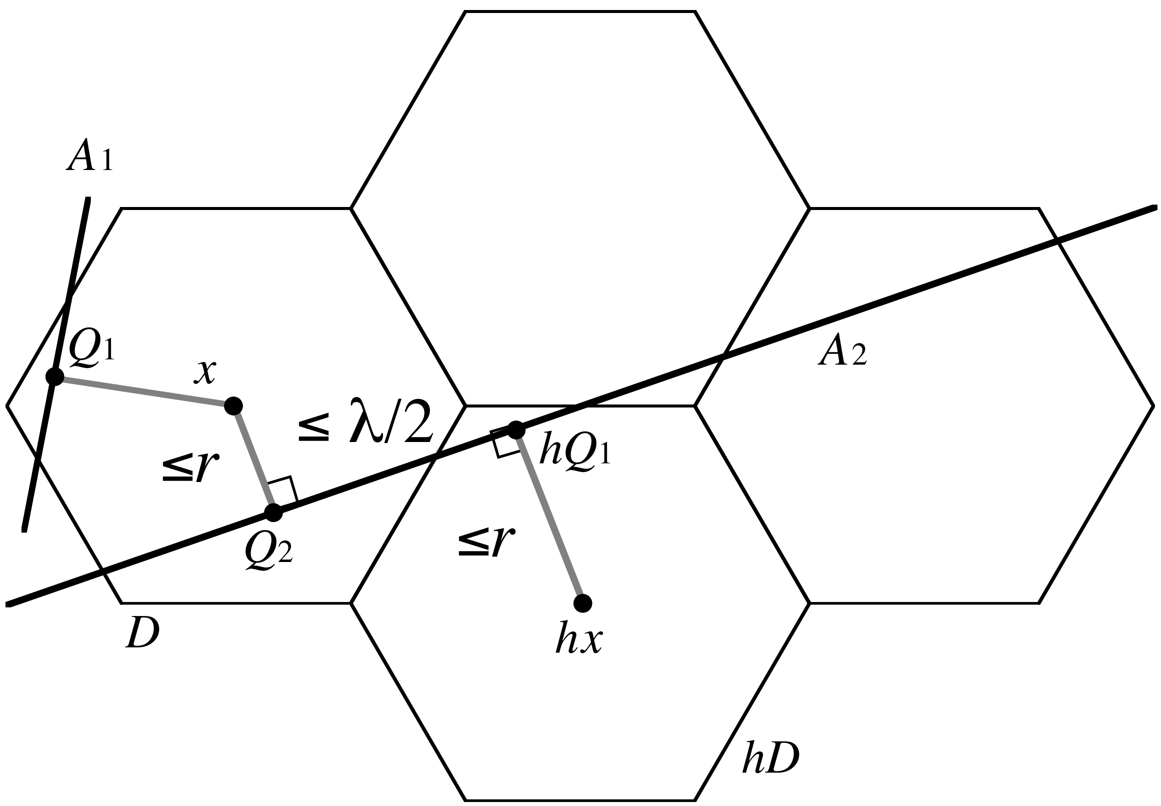


Figure 1.3. The distance from x to hx is at most 2 $\cosh^{-1}(\cosh(r) \cosh(\lambda/4))$.

Proposition 1.7. If g_1 and g_2 are conjugate group elements corresponding to a geodesic of length at most λ, and both their axes pass within a distance r of the basepoint, then there is a group element h such that

(1) $g_1 = h\, g_2\, h^{-1}$ and

(2) $d(x, hx) \leq 2 \cosh^{-1}(\cosh(r) \cosh(\lambda/4))$.

Proof. Let A_i be the axis of g_i, and Q_i be the point on A_i closest to the basepoint x. (See Figure 1.3.) There are infinitely many covering transformations taking A_1 to A_2; let h be one which minimizes the distance from hQ_1 to Q_2. Because the length of the underlying geodesic is at most λ, the distance from hQ_1 to Q_2 is at most $\lambda/2$ (it's $\lambda/2$ and not λ because if $\lambda/2 < d(hQ_1, Q_2) < \lambda$ then you've got the wrong h — you need to consider an h which takes Q_1 to a point on the other side of Q_2). Imitate the proof of Proposition 1.6.2 to get the desired bound $d(x, hx) \leq 2\cosh^{-1}(\cosh(r)\cosh(\lambda/4))$. *Q.E.D.*

Comment. The bound on d(x, hx) in Proposition 1.7 is less than the bound on d(x, gx) in Proposition 1.6.2. This implies that the big list contains all the conjugating matrices we could possibly need in Step 4 of the above algorithm.

This concludes the discussion of the algorithm for rigorously computing a length spectrum with multiplicities.

2. Examples. Knowing the lengths and multiplicities of the shortest geodesics, we may now apply the technique outlined in the introduction to compute the symmetry groups of the smallest known closed hyperbolic 3-manifolds. These manifolds are part of a census of closed hyperbolic 3-manifolds which was obtained by Dehn filling on a census of cusped hyperbolic 3-manifolds [HiW]. A full report on the census of closed manifolds will appear in [HoW].

Table 1 below gives a symmetric surgery description for each manifold, and lists the manifold's volume, Chern-Simons invariant (mod□1/2), first homology group, symmetry group, and length spectrum to $\lambda = 2.0$. All the smallest known hyperbolic 3-manifolds are orientable.

Table 2 provides examples of closed nonorientable hyperbolic 3-manifolds. Unlike the manifolds in Table 1, which come from extensive and careful computer searches, the nonorientable manifolds in Table 2 come from a quick preliminary search for closed nonorientable manifolds. Yet two observations are striking. First, the smallest closed nonorientable manifold we have found has volume 2.0298…. Why should it be so much larger than both the smallest known closed orientable manifold (volume 0.9427…) and the smallest known cusped nonorientable manifold (volume 1.0149…) ? Second, most of the low-volume closed nonorientable manifolds we found have the same volumes as the 1-cusp manifolds obtained by Dehn filling one component of the Whitehead link complement. It turns out that these manifolds are obtainable as Dehn fillings on an apparent mutant of the Whitehead link complement. The mutant (denoted *WLM* in Table 2) has an orientable cusp — thereby allowing an infinite number of distinct Dehn fillings — even though the manifold as a whole is nonorientable. Not only do the volumes of the closed manifolds obtained by Dehn filling on *WLM* coincide with the volumes of the cusped manifolds obtained by Dehn filling (one cusp of) the Whitehead link complement, the complex lengths of the surgery curves also coincide exactly. The authors plan to discuss *WLM* at greater length in [HoW].

Comment. E. Molnár computes the symmetry group of the closed manifold of volume 0.94… as the group of outer automorphisms of its fundamental group [M]. He finds it to be $Z_2 \times D_3 \approx D_6$ in agreement with SnapPea's calculation.

Extended comment. *Negatively oriented tetrahedra*. The algorithm for drilling out closed geodesics may sometimes drill out a set L' of simple closed curves which is homotopic but not isotopic to the intended geodesic link L. In these cases the computed symmetry group will be a subgroup of the true symmetry group. Roughly speaking, every symmetry of $M - L'$ which preserves meridians on the boundary extends to a symmetry of M, but not every symmetry of M restricts to a symmetry of $M - L'$. More precisely, we have the following proposition.

Proposition 2.1. If L' is a set of nonintersecting (not necessarily geodesic) simple closed curves in M, and $M - L'$ is hyperbolic, then the subgroup of $\mathrm{Isom}(M - L')$ which sends meridians to meridians injects naturally into $\mathrm{Isom}(M)$.

Proof. This follows from an observation of Sakuma, quoted in the proof of Proposition 6 in [K]. Borel's theorem (see [CR]) states that if G is a finite group acting effectively on a closed aspherical manifold M and $\pi_1(M)$ has trivial center, then the induced homomorphism from G to Out $\pi_1(M)$ is injective. So let G be the subgroup of $\mathrm{Isom}(M - L')$ which sends meridians to meridians, and then apply Mostow rigidity to equate Out $\pi_1(M)$ with $\mathrm{Isom}(M)$. *Q.E.D.*

According to Thurston's theory of hyperbolic Dehn filling ([Th], Ch. 4), if we can recover the hyperbolic structure on M as the metric completion of an incomplete hyperbolic structure on $M - L'$ triangulated by positively oriented ideal tetrahedra, then the link L' must be isotopic to a geodesic link L. So whenever SnapPea finds a solution with all positively oriented tetrahedra, we know we have drilled out the intended geodesic link L, and we may apply Proposition 0.2 to compute the symmetry group of M. But when the solution contains negatively oriented tetrahedra, we can't be sure whether the link L' is isotopic to a geodesic link L or not. The best we can do is apply Proposition 2.1 to compute a group which may be either $\mathrm{Isom}(M)$ or a proper subgroup of $\mathrm{Isom}(M)$. Fortunately solutions with all positively oriented tetrahedra may be found for the vast majority of closed manifolds, and in these cases the symmetry groups may be computed exactly.

The only low-volume manifolds for which the authors have failed to find a hyperbolic Dehn filling with all positively oriented tetrahedra are the manifolds of volume 0.94…, 1.01… and 1.26…. For example, if M is the manifold of volume 0.94… and L is the set of three geodesics of length 0.58…, then we obtain the surgery diagram shown in Table 1, which suggests the symmetry group of M is D_6. But the Dehn filling on $M - L$ creates negatively oriented tetrahedra (in the given triangulation), so this evidence suffices only to conclude that the symmetry group *contains* D_6, not that the symmetry group *is* D_6. Fortunately the following additional evidence shows rigorously that the symmetry group really is D_6. If we drill out a single curve γ of length 0.58…, we get a description of M as a Dehn filling on a 1-cusp manifold. In this case the hyperbolic Dehn filling keeps all tetrahedra positively oriented, so we know the curve we drilled out is isotopic to the geodesic γ. We apply Proposition 2.2 to conclude that the group of symmetries of M fixing γ is isomorphic to the subgroup of Isom($M - \gamma$) which preserves meridians. SnapPea computes the latter group to be D_2. We also know that M contains precisely three geodesics of length 0.58…. It follows that the full symmetry group of M can have at most $3 \times 4 = 12$ elements. Therefore the symmetry group of M is exactly D_6.

Similar evidence shows that the symmetry group of the manifold of volume 1.26… is exactly D_4.

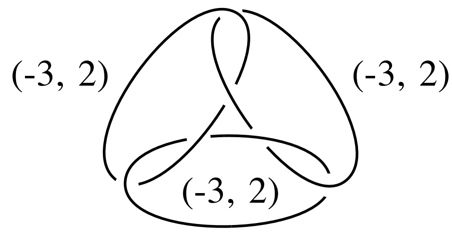
All known surgery descriptions for the manifold of volume 1.01… involve negatively oriented tetrahedra, so here we used an ad hoc technique to verify the symmetry group computation. We computed a fundamental domain for this manifold and then manually drilled out the unique geodesic of length 1.31… (the old-fashioned way, with pencil and paper) to check that its complement was the same as the one SnapPea had found.

Proposition 2.2 (Kojima [K])**.** If L is a (not necessarily complete) simple geodesic link in a closed hyperbolic 3-manifold M, then the subgroup of Isom(M) which preserves L is naturally isomorphic to the subgroup of Isom($M - L$) which takes meridians to meridians.

Proof. This proposition is a generalization of Proposition 0.2. The proof follows immediately from Lemma 5 and Proposition 6 of [K]. *Q.E.D.*

Table 1. *The ten smallest known closed hyperbolic 3-manifolds*. Each surgery description embodies the manifold's full symmetry group; the lengths of the core curves are marked with asterisks in the accompanying length spectrum. The Chern-Simons invariant is given mod 1/2. The symbol D_n denotes the dihedral group of order $2n$. The length spectra were computed to $\lambda = 2.0$. On a Macintosh SE/30 each length spectrum computation took about 30 seconds. The other computations were almost instantaneous.

#1



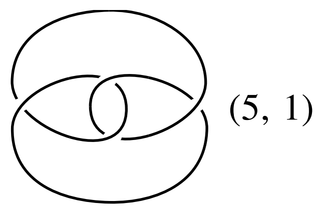
vol = 0.942707362776928 CS = 0.060043066678727

$H_1 = Z_5 + Z_5$ Isom = D_6

length spectrum:

multiplicity	length			torsion	
3	0.5846036850179870	+	i	2.4953704555604684	*
3	0.7941346629918657	-	i	2.3048568167434712	
3	1.2898511615162902	+	i	2.4839246176639380	

#2



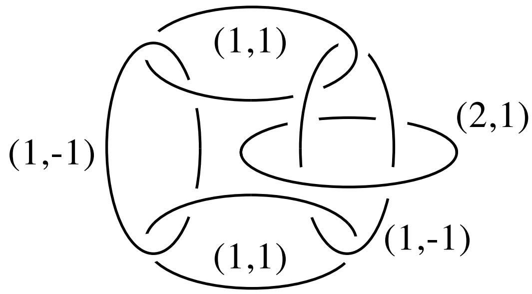
vol = 0.981368828892232 CS = 0.077038180263770

$H_1 = Z_5$ Isom = D_2

length spectrum:

multiplicity	length			torsion	
1	0.5780824355318555	+	i	2.1324306361541782	
1	0.7215683663144467	-	i	1.1512129898755381	*
2	0.8894429972125508	+	i	2.9418590470227377	
2	0.9983251885367372	-	i	2.9210177857853122	
1	1.0403151250687664	+	i	0.9823718902812068	
2	1.7938008425243408	-	i	1.5568710482226091	
1	1.8222796995026176	-	i	2.4135390256883349	

#3

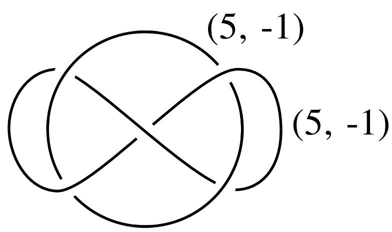


(This surgery description displays only the orientation preserving symmetries. To see the orientation reversing symmetries, do a full twist on the (2,1) component and take the mirror image.)

vol = 1.014941606409654 CS = 0.000000000000000
H_1 = $Z_3 + Z_6$ Isom = semidihedral of order 16
length spectrum:

multiplicity	length		torsion	
2	0.8314429455293105	-	i 1.9455307595036367	
2	0.8314429455293105	+	i 1.9455307595036367	
2	0.8625546276620610	-	i 2.6806731878491248	*
2	0.8625546276620610	+	i 2.6806731878491248	*
1	1.3169578969248167	+	i 3.1415926535897932	*
2	1.9916523910494368	-	i 1.2995325730688758	
2	1.9916523910494368	+	i 1.2995325730688758	

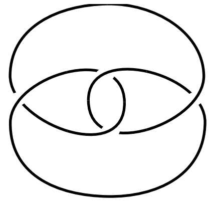
#4



vol = 1.263709238658044 CS = 0.114140655969370
H_1 = $Z_5 + Z_5$ Isom = D_4
length spectrum:

multiplicity	length		torsion	
2	0.5750785808501463	+	i 1.7453223490489995	*
2	0.6668359270289546	-	i 2.8250337474759131	
4	1.1685811002431094	+	i 2.6310591157182455	
2	1.8042180449094749	-	i 1.5223476911464933	
2	1.8159868034200992	+	i 2.2362704849933792	

#5



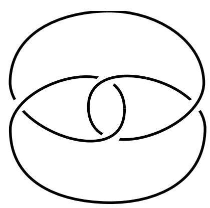
vol = 1.284485300468354 CS = 0.067931673479863

H_1 = Z_6 Isom = D_2

length spectrum:

multiplicity	length			torsion	
1	0.4803118033850776	-	i	0.9337766566662489	*
1	0.7575722478871622	+	i	2.0617461176175744	
2	1.1487060968062998	+	i	2.8549051411832343	
1	1.2669264000810559	+	i	1.0280870933914297	
2	1.3653169486199054	-	i	2.9588409030245944	
1	1.4409354101552327	-	i	2.8013299699987466	
1	1.5151444957743244	-	i	2.1596930719444376	
1	1.5618887811062861	+	i	1.2110739839018146	
1	1.9065758281061861	-	i	1.4536426123187809	

#6



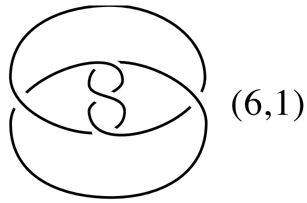
vol = 1.398508884150807 CS = -0.246607252650769

H_1 = trivial Isom = D_2

length spectrum:

multiplicity	length			torsion	
1	0.3661307023464804	+	i	3.1135899402525185	*
1	0.9068955815336451	+	i	1.3522227290734487	
1	1.0729359343958735	-	i	1.2904875910594336	
2	1.4390666367423539	+	i	1.8231023491930849	
2	1.6923383304376530	-	i	1.7538512695256377	
1	1.7797411664452666	+	i	0.5886201848082008	
1	1.9041599087594012	-	i	0.6237663509730675	

#7



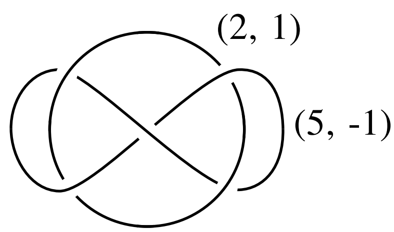
vol = 1.414061044165392 CS = -0.076602066648576

H_1 = Z_6 Isom = D_2

length spectrum:

multiplicity	length			torsion	
1	0.7941346629918657	-	i	2.3048568167434712	
1	0.8435987229688855	+	i	1.4077154426029530	
1	0.8539717590561439	-	i	0.8860692766020361	*
1	1.0895750721306458	+	i	1.7011475160827151	
1	1.1895376734877818	-	i	2.7548032918340546	
2	1.2328780837767570	-	i	2.9615306562993910	
2	1.3518274333239401	+	i	2.7156753611675211	
1	1.9623642391042593	-	i	0.6617663270266211	

#8



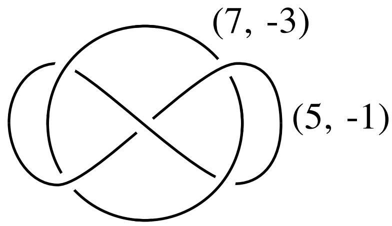
vol = 1.414061044165392 CS = -0.118268733315243

H_1 = Z_{10} Isom = D_2

length spectrum:

multiplicity	length			torsion	
1	0.3648946865259024	-	i	1.6282435124992135	*
1	0.8435987229688855	+	i	1.4077154426029530	*
2	0.8580212576166383	-	i	2.8466755988176295	
2	0.9811821195521296	+	i	2.8107094900764827	
1	1.1692073700359739	-	i	1.2924443960586497	
1	1.7079435181122878	+	i	1.3694541003857210	
1	1.8212713005129035	-	i	0.5179479759806375	
2	1.8659841906480162	+	i	2.1104546278259760	

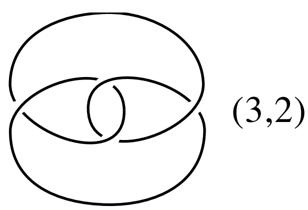
#9



vol = 1.423611900292825 CS = -0.112565152046181
H_1 = Z_{35} Isom = D_2
length spectrum:

multiplicity	length		torsion	
1	0.3523715984106593	-	i 1.7555306580636763	*
2	0.7221944677176101	-	i 2.5616046010833886	
2	0.8324780979472408	+	i 2.4871598131039233	
1	0.9613222986604775	+	i 2.0832746891099856	*
1	1.2237734200522688	-	i 2.1175419864720803	
2	1.7440957764400803	-	i 2.8056186310694022	
2	1.7503932130104359	-	i 2.2549932108325395	
2	1.8608290851883280	+	i 1.7943024301306726	

#10



vol = 1.440699006727365 CS = -0.240060709461803
H_1 = Z_3 Isom = D_2
length spectrum:

multiplicity	length		torsion	
1	0.3615215761199664	+	i 3.0518203522989750	*
1	0.7966230904490147	+	i 1.4750598219976215	
2	1.2589722309329429	+	i 1.9929928856385542	
1	1.2670535807636463	-	i 1.3227078914922182	
1	1.6527923153487228	+	i 0.6045557843178742	
2	1.7392943589048573	+	i 2.2138108352220500	
2	1.9721776145120246	-	i 1.8248162737791342	

Table 2. *Low-volume closed nonorientable hyperbolic 3-manifolds*. Unlike the results in Table 1, which are backed by extensive and careful computer searches, the manifolds in this table should be considered only tentative

candidates for the smallest closed nonorientable hyperbolic 3-manifolds. All happen to be Dehn fillings on an apparent mutant of the Whitehead link complement, denoted *WLM*. *WLM* is nonorientable but has an orientable cusp. The cusp shape is a 2:1 rectangle (i.e. its conformal invariant is $2i$); the Dehn fillings in the table are expressed relative to the basis (short side, long side). Torsion values are given for orientation preserving geodesics only, since the notion of torsion does not apply to orientation reversing ones. Note that manifolds #8 and #9 have the same volume and first homology group as the cusped manifolds *WL*(4,1) and *WL*(4,-3) which are discussed at length in [HMW].

#1 *WLM*(3, 1)

vol = 2.029883212819307

H_1 = Z   Isom = Z_2

length spectrum:

multiplicity	length			torsion	
1	0.4812118250596034				
1	1.0870701449957391	+	i	1.7227684498700903	*
3	1.4436354751788106				
1	1.6628858910586211	+	i	2.3921237881723131	
4	1.8317806070782301	+	i	2.3497882674791975	
1	1.9649783751106262	+	i	1.7086915352390857	

#2 *WLM*(1, 2)

vol = 2.568970600936709

H_1 = Z   Isom = Z_2

length spectrum:

multiplicity	length			torsion	
1	0.6545044131847577	+	i	3.0440377289969229	*
1	0.7555258440155862				
1	1.0195337194432843				
2	1.3509265391467243	+	i	1.5957466532081133	
2	1.5713677340101971				
1	1.6774723007382255	+	i	2.4487416724056426	
2	1.9587222861663571				

#3 *WLM*(4, 1)

vol = 2.666744783449060

H_1 = Z + Z_2 Isom = Z_2

length spectrum:

multiplicity	length			torsion	
1	0.6329743192009473	+	i	1.3621796364857481	*
1	0.6584789484624083				

1	1.1462158347805888			
2	1.6086359855786289			
2	1.6723527369735115	+	i 2.2587043011101221	
1	1.7159328483907753	+	i 2.6276818911744879	
1	1.9080635728244040	+	i 0.6460888340900950	
2	1.9754368453872251			

#4 *WLM*(3, 2)

vol = 2.828122088330783

H_1 = Z Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.5623991486459237	+	i 2.8154308852059061	*
1	0.6215573417511161			
1	1.2001623484845858			
2	1.5391837346884633	+	i 1.8395768183633292	
1	1.7533180991533151	+	i 2.9107044294647036	
2	1.7664053409301833			
2	1.8646720252533484			

#5 *WLM*(5, 1)

vol = 2.989120282929485

H_1 = Z Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.4234738239334289	+	i 1.1300530516785054	*
1	0.7453207268602946			
1	1.0319358685218621			
2	1.6201166033782924	+	i 2.1949633051730797	
1	1.7377432516544678	+	i 2.7657773826850275	
2	1.7672280816572518			

#6 *WLM*(1, 3)

vol = 3.148509826440728

H_1 = Z Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.3169358116334042	+	i 2.0530607878187304	*
1	0.8327431296509962			
1	0.9317362138037721			
2	1.4451333651900341	+	i 1.8071533025565227	
1	1.7423360741844900	+	i 2.8019975741924283	
2	1.8844017437091807			

#7 *WLM*(5, 2)

vol = 3.163963228883144

H_1 = Z   Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.3306355216310498	+ i	2.7451641826568332	*
1	0.7154087390876620			
1	1.0694699418658219			
2	1.5737804919308885	+ i	2.0406097338944055	
1	1.7611378592250281	+ i	3.0461840765608912	
2	1.9035091919294842			

#8 *WLM*(6, 1)

vol = 3.177293278600326

H_1 = Z + Z_4 Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.3046889317180031	+ i	0.9653591080979228	*
1	0.7928356308754387			
1	0.9758291292144478			
2	1.5944792705564403	+ i	2.1502880719367992	
1	1.7482859664020846	+ i	2.8556982395731824	
2	1.8697359017008077			

#9 *WLM*(2, 3)

vol = 3.177293278600326

H_1 = Z + Z_4 Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.3046889317180031	+ i	2.1762335454918704	*
1	0.7928356308754387			
1	0.9758291292144478			
2	1.4644960509912102	+ i	1.8353604843400738	
1	1.7482859664020846	+ i	2.8556982395731824	
2	1.8697359017008077			

#10 *WLM*(4, 3)

vol = 3.275871643943934

H_1 = Z + Z_2 Isom = Z_2

length spectrum:

multiplicity	length		torsion	
1	0.2522612564171241	+ i	1.9456904341156199	*
1	0.7606180249800409			

1	1.0134191209918691			
2	1.5127001446969349	+	i 1.9257185935315922	
1	1.7601756393679013	+	i 3.0209916805119804	
2	1.9298219664372945			

Table 3. *Hyperbolic 3-orbifolds.* SnapPea uses the results of §1 to compute length spectra for orbifolds as well as manifolds, but cannot in general compute the symmetry group of an orbifold. Here are length spectra for three sample orbifolds. The symbol *mI* indicates that a geodesic is topologically a mirrored interval, not a circle. The length spectra for orbifolds #1 and #3 are shown to $\lambda = 2.0$; for #2 it's shown to $\lambda = 3.0$.

#1 $\mathbf{H}^3 / PSL(2, \mathbf{Z}[i])$ (generators courtesy of Pat Callahan)

multiplicity	length		torsion	
2	0.9624236501192069	+	i 3.1415926535897932	mI
2	1.0612750619050356	-	i 2.2370357592874119	mI
2	1.0612750619050356	+	i 2.2370357592874119	mI
1	1.4657153519472905	-	i 1.3497776911720128	mI
1	1.4657153519472905	+	i 1.3497776911720128	mI
3	1.7627471740390861	+	i 3.1415926535897932	mI
2	1.8522660627003648	-	i 2.4427145278753667	mI
2	1.8522660627003648	+	i 2.4427145278753667	mI

#2 mirrored regular ideal tetrahedron

multiplicity	length		torsion	
3	2.1741402899914782	+	i 2.8376484074394059	
6	2.3895264345742186			
4	2.5876638829861831	+	i 1.7588342563677879	
3	2.6339157938496334	+	i 0.0000000000000000	mI
6	2.9215632262264709	+	i 1.3817440905462071	
12	2.9932228461263809			

#3 FigureEight(4, 0)

multiplicity	length		torsion
1	0.8625546276620610	-	i 2.6806731878491248
1	0.8625546276620610	+	i 2.6806731878491248
1	1.0870701449957391	-	i 1.7227684498700903
1	1.0870701449957391	+	i 1.7227684498700903

2	1.2080566434549810	-	i 2.5372497569980431	
2	1.2080566434549810	+	i 2.5372497569980431	
1	1.6628858910586211	-	i 2.3921237881723131	mI
1	1.6628858910586210	+	i 2.3921237881723131	mI
1	1.7627471740390860	+	i 0.0000000000000000	

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