NOTES ON RICHARD THOMPSON'S GROUPS F AND T

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The groups F and T were defined by Richard Thompson in 1965. McKenzie and Thompson used F and T in [M-T] to construct finitely-presented groups with unsolvable word problems. In unpublished notes, Thompson proved that T is a finitely-presented, infinite simple group. The group F was rediscovered by homotopy theorists in connection with work on homotopy idempotents (see [D1], [D2] and [F-H]). Partly motivated by the connection with homotopy theory, Brown and Geoghegan [B-G] proved that F is FP_∞, thereby giving an example of a torsion-free infinite-dimensional FP_∞ group. These groups (and generalizations of them) continue to be of interest; however, we will not attempt here to describe all of the known results about them.

This paper is largely expository, and much of the material in it is standard. Since it is not readily available, we have included Thompson's unpublished proof that T is simple. We have also included proofs that $F \cong PIP^+([0,1])$ and $T \cong PIP^+(S^1)$.

In §1 we define F as a group of piecewise linear homeomorphisms of the unit interval $[0,1]$, and then give some examples of elements of F. In §2 we represent elements of F as tree diagrams, and give a normal form for elements of F. Two standard presentations for F are given in §3. In §4 we prove several theorems about F; these are partly motivated by the question of whether F is an amenable group. In §5 we define T and give Thompson's unpublished proof that T is simple. In §6 we give W. Thurston's interpretations of F and T in terms of piecewise integral projective homeomorphisms of the interval and the circle.

The group that we are denoting F was originally denoted $\widehat{\mathbb{P}}$ in Thompson's unpublished notes and $\mathfrak{P}'$ in [M-T], and was denoted $\mathbb{P}$ in [T]. It was denoted F in [B-G] in 1984, and it was also denoted F in [B-S], [B1], [B3], [Ge], [G-S], [Gr], [Gr-S] and [S]. It is denoted G in [B-S].

The group that we are denoting T was originally denoted $\widehat{\mathcal{C}}$ in Thompson's unpublished notes. It was denoted T in [B1] in 1987 and was denoted T in [S]. However, it was denoted G in [G-S] and [Gr]. It is denoted S in [B-S].

There is a third group defined by Thompson that corresponds (in our notation) to the group of piecewise dyadic bijections of the unit interval. This group contains F and T as subgroups, and is a finitely-presented, infinite simple group. It was originally denoted $\widehat{V}$ in Thompson's unpublished notes and $\mathfrak{C}'$ in [M-T], and was denoted $Ft({}^\omega 2)$ in [T]. It was denoted $G_{2,1}$ in [H] in 1974, and was denoted G in [B1], [B2] and [S].

While we have not attempted to survey the literature on Thompson's groups F and T, we have included in the bibliography those references of which we are aware.

§1. Introduction to F.

Let F be the set of orientation-preserving piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2. Let $f \in F$, and let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be the points at which f is not differentiable. Then since $f(0) = 0$, $f(x) = a_1 x$ for $x_0 \le x \le x_1$, where a_1 is a power of 2. Likewise, since $f(x_1)$ is a dyadic rational number, $f(x) = a_2 x + b_2$ for $x_1 \le x \le x_2$, where a_2 is a power of 2 and b_2 is a dyadic rational number. It follows inductively that $f(x) = a_i x + b_i$ for $x_{i-1} \le x \le x_i$ and $i = 1, \dots, n$, where a_i is a power of 2 and b_i is a dyadic rational number. It easily follows that $f^{-1} \in F$ and that f maps the set of dyadic rational numbers bijectively to itself. From this it is easy to see that F is closed under composition of functions. Thus F is a subgroup of the group of all homeomorphisms from $[0,1]$ to $[0,1]$. This group F is Thompson's group F.

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EXAMPLE 1.1. Two functions in F are the functions A and B defined below.

$$A(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \le x \le 1 \end{cases} \qquad B(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \le x \le 1 \end{cases}$$

A useful notation for functions f in F will be described next. Construct a rectangle with a top, which is viewed as the domain of f, and a bottom, which is viewed as the range of f. For every point x on the top where f is not differentiable, construct a line segment from x to $f(x)$ on the bottom. Call the result the *rectangle diagram* of f. By juxtaposing the rectangle diagrams of a pair of functions, it is easy to compute the rectangle diagram of their composition. We learned about rectangle diagrams from W. Thurston in 1975; they also appear in [B-S].

EXAMPLE 1.2. Figure 1 gives some examples of functions in F and their rectangle diagrams.

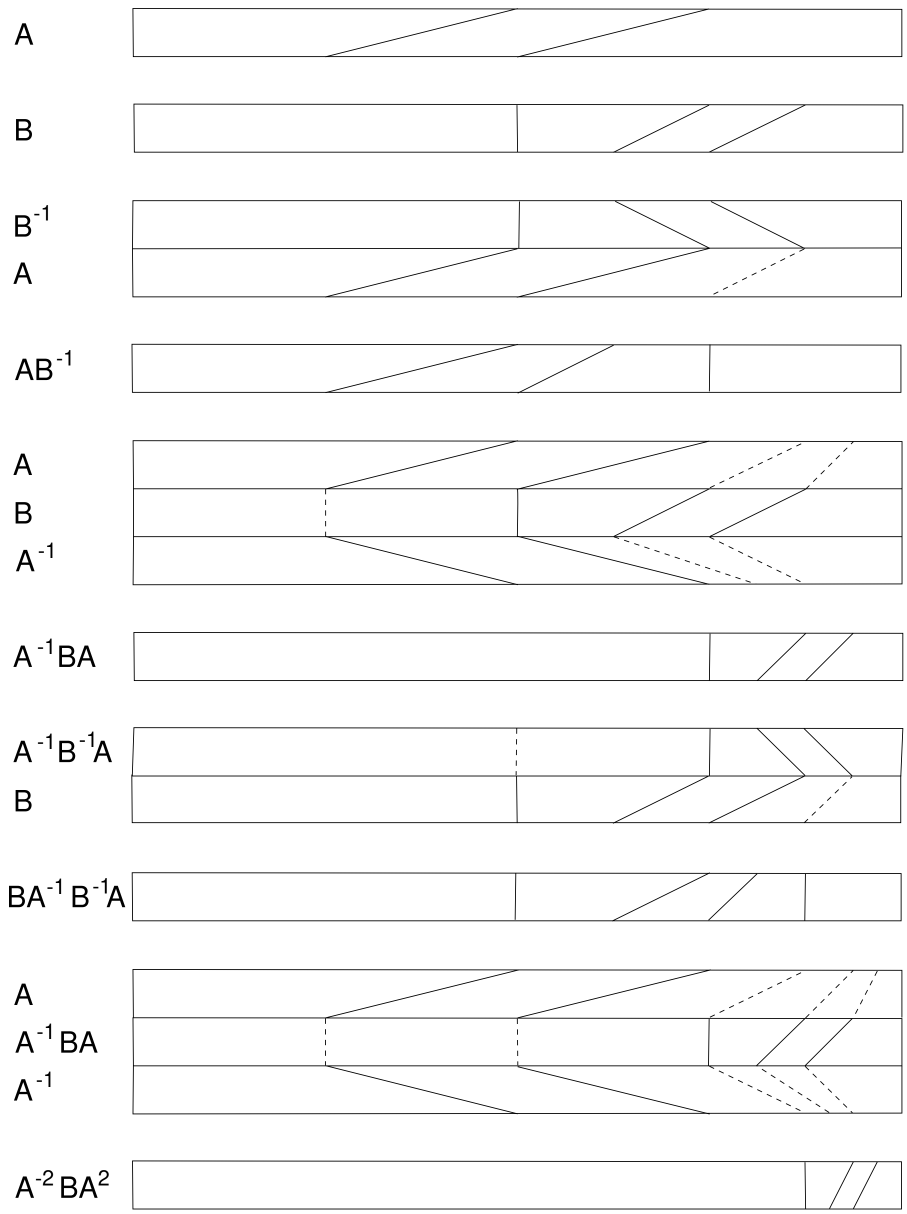

Figure 1. The rectangle diagrams of some elements of F.

Define functions $X_0, X_1, X_2, \ldots$ in F so that $X_0 = A$ and $X_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$. From Example 1.2 it is easy to see that the rectangle diagram of X_n is as in Figure 2.

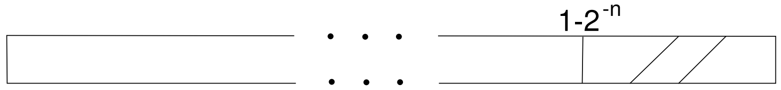

Figure 2. The rectangle diagram of X_n.

§2. Tree diagrams.

The notion of tree diagram is developed in this section. Tree diagrams are useful for describing functions in F; we first encountered them in [B1].

Define an *ordered rooted binary tree* to be a tree S such that i) S has a root v_0, ii) if S consists of more than v_0, then v_0 has valence 2 and iii) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v and are not contained in the geodesic from v_0 to v. The edge $e_{v,L}$ is called a *left edge* of S, and $e_{v,R}$ is called a *right edge* of S. Vertices with valence 0 (in case of the trivial tree) or 1 in S will be called *leaves* of S. There is a canonical left-to-right linear ordering on the leaves of S. The *right side* of S is the maximal path of right edges in S which begins at the root of S. The *left side* of S is defined analogously.

An isomorphism of ordered rooted binary trees is an isomorphism of rooted trees which takes left edges to left edges and right edges to right edges. An ordered rooted binary subtree S' of an ordered rooted binary tree S is an ordered rooted binary tree which is a subtree of S whose left edges are left edges of S, whose right edges are right edges of S, but whose root need not be the root of S.

Example 2.1. The right side of the ordered rooted binary tree in Figure 3 is highlighted. Its leaves are labeled $0, \ldots, 5$ in order.

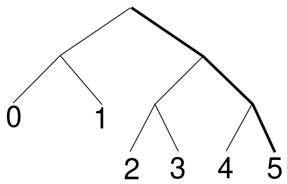

Figure 3. An ordered rooted binary tree with 6 leaves.

Define a *standard dyadic interval* in $[0,1]$ to be an interval of the form $[\frac{a}{2^n}, \frac{a+1}{2^n}]$, where a, n are nonnegative integers with $a \leq 2^n - 1$.

There is a *tree of standard dyadic intervals*, $\mathcal{T}$, which is defined as follows. The vertices of $\mathcal{T}$ are the standard dyadic intervals in $[0,1]$. An edge of $\mathcal{T}$ is a pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I, J) is a left edge, or I is the right half of J, in which case (I, J) is a right edge. It is easy to see that $\mathcal{T}$ is an ordered rooted binary tree. The tree of standard dyadic intervals is shown in Figure 4.

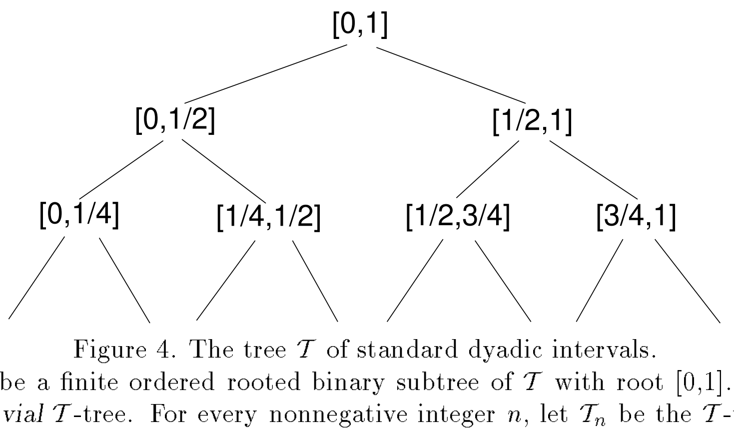

Figure 4. The tree $\mathcal{T}$ of standard dyadic intervals.

Define a *$\mathcal{T}$-tree* to be a finite ordered rooted binary subtree of $\mathcal{T}$ with root [0,1]. Call the $\mathcal{T}$-tree with just one vertex the *trivial* $\mathcal{T}$-tree. For every nonnegative integer n, let $\mathcal{T}_n$ be the $\mathcal{T}$-tree with $n+1$ leaves whose right side has length n. $\mathcal{T}_3$ is shown in Figure 5.

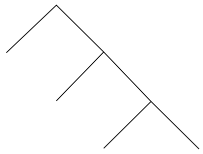

Figure 5. The $\mathcal{T}$-tree $\mathcal{T}_3$.

Define a *caret* to be an ordered rooted binary subtree of $\mathcal{T}$ with exactly two edges. Every caret has the form of the rooted tree in Figure 6.

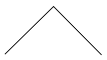

Figure 6. A caret.

A partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ of $[0, 1]$ determines intervals $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$ which are called the *intervals of the partition*. A partition of [0,1] is called a *standard dyadic partition* if and only if the intervals of the partition are standard dyadic intervals.

It is easy to see that the leaves of a $\mathcal{T}$-tree are the intervals of a standard dyadic partition. Conversely, the intervals of a standard dyadic partition determine finitely many vertices of $\mathcal{T}$, and it is easy to see that these vertices are the leaves of their convex hull, which is a $\mathcal{T}$-tree. Thus there is a canonical bijection between standard dyadic partitions and $\mathcal{T}$-trees.

Lemma 2.2. *Let $f \in F$. Then there exists a standard dyadic partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that f is linear on every interval of the partition and $0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = 1$ is a standard dyadic partition.*

Proof. Choose a partition P of $[0, 1]$ whose partition points are dyadic rational numbers such that f is linear on every interval of P. Let $[a, b]$ be an interval of P. Suppose that the derivative of f on $[a, b]$ is 2^{-k}. Let n be an integer such that $2^n a \in \mathbf{Z}$, $2^n b \in \mathbf{Z}$, $2^{n+k} f(a) \in \mathbf{Z}$ and $2^{n+k} f(b) \in \mathbf{Z}$. Then $a < a + \frac{1}{2^n} < a + \frac{2}{2^n} < a + \frac{3}{2^n} < \cdots < b$ partitions $[a, b]$ into standard dyadic intervals, and $f(a) < f(a) + \frac{1}{2^{n+k}} < f(a) + \frac{2}{2^{n+k}} < f(a) + \frac{3}{2^{n+k}} < \cdots < f(b)$ partitions $[f(a), f(b)]$ into standard dyadic intervals. This easily proves Lemma 2.2. ∎

Formally, a *tree diagram* is an ordered pair (R, S) of $\mathcal{T}$-trees such that R and S have the same number of leaves. This is rendered diagrammatically as follows:

$$R \to S.$$

The tree R is called the *domain tree* of the diagram, and S is called the *range tree* of the diagram.

Suppose given $f \in F$. Lemma 2.2 shows that there exist standard dyadic partitions P and Q such that f is linear on the intervals of P and maps them to the intervals of Q. To f is associated the tree diagram (R, S), where R is the $\mathcal{T}$-tree corresponding to P and S is the $\mathcal{T}$-tree corresponding to Q.

Because P and Q are not unique, there are many tree diagrams associated to f. Given one tree diagram (R, S) for f, another can be constructed by adjoining carets to R and S as follows. Let I be the n^{th} leaf of R for some positive integer n, and let J be the n^{th} leaf of S. Let I_1, I_2 be the leaves in order of the caret C with root I, and let J_1, J_2 be the leaves in order of the caret D with root J. Because f is linear on I and $f(I) = J$, it follows that $f(I_1) = J_1$ and $f(I_2) = J_2$. Thus (R', S') is a tree diagram for f, where $R' = R \cup C$ and $S' = S \cup D$.

In the other direction, if there exists a positive integer n such that the n^{th} and $(n + 1)^{\text{th}}$ leaves of R, respectively S, are the vertices of a caret C, respectively D, then deleting all but the roots of C and D from R and S leads to a new tree diagram for f. If there do not exist such carets C, D in R, S, then the tree diagram (R, S) is said to be *reduced*.

In this paragraph it will be shown that there is exactly one reduced tree diagram for f. Suppose that (R, S) is a reduced tree diagram for f. It is easy to see that if I is a standard dyadic interval which is either a leaf of R or not in R, then $f(I)$ is a standard dyadic interval and f is linear on I. Conversely, if I is a standard dyadic interval such that $f(I)$ is a standard dyadic interval and f is linear on I, then I is either a leaf of R or not in R because (R, S) is reduced. Thus R is the unique $\mathcal{T}$-tree such that a standard dyadic interval I is either a leaf of R or not in R if and only if $f(I)$ is a standard dyadic interval and f is linear on I. This gives uniqueness of reduced tree diagrams.

Furthermore, if (R, S) is a tree diagram, then it is clear that there exists $f \in F$ such that f is linear on every leaf of R and f maps the leaves of R to the leaves of S.

Thus there is a canonical bijection between F and the set of reduced tree diagrams.

Example 2.3. Figure 7 shows the reduced tree diagrams for A and B.

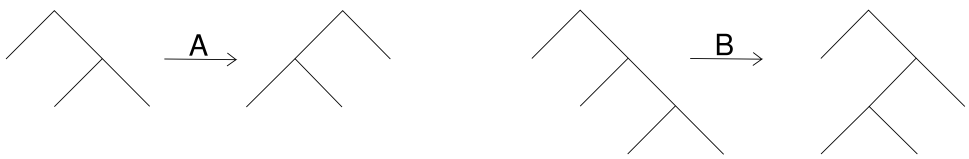

Figure 7. The reduced tree diagrams for A and B.

From Figure 2 it is not difficult to see that, for $n \geq 0$, the reduced tree diagram for X_n is the tree diagram in Figure 8.

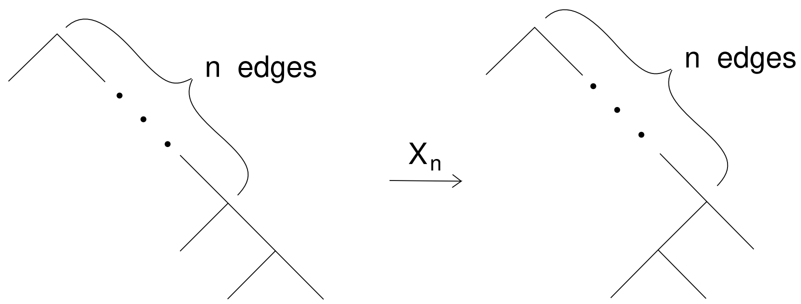

Figure 8. The reduced tree diagram for X_n.

It is easy to see that if (Q, R) is a tree diagram for a function f in F and (R, S) is a tree diagram for a function g in F, then (Q, S) is a tree diagram for gf.

The following definition prepares for Theorem 2.5, which makes the correspondence between functions in F and tree diagrams more precise. Define the *exponents* of a $\mathcal{T}$-tree S as follows. Let $I_0, \ldots, I_n$ be the leaves of S in order. For every integer k with $0 \leq k \leq n$ let a_k be the length of the maximal path of left edges in S which begins at I_k and which does not reach the right side of S. Then a_k is the k^{th} *exponent* of S.

Example 2.4. Let S be the $\mathcal{T}$-tree shown in Figure 9.

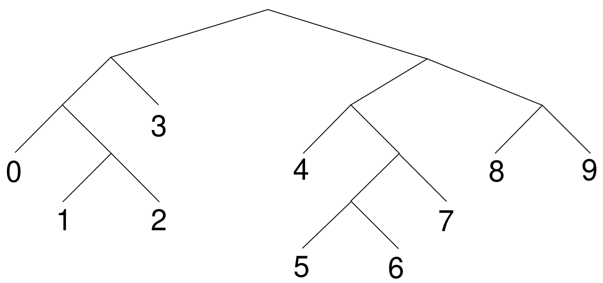

Figure 9. A $\mathcal{T}$-tree.

The leaves of S are labeled $0, \ldots, 9$ in order, and the exponents of S in order are 2,1,0,0,1,2,0,0,0,0.

Theorem 2.5. *Let R, S be $\mathcal{T}$-trees with $n+1$ leaves for some nonnegative integer n. Let $a_0, \ldots, a_n$ be the exponents of R, and let $b_0, \ldots, b_n$ be the exponents of S. Then the function in F with tree diagram (R, S) is $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$. The tree diagram (R, S) is reduced if and only if i) if the last two leaves of R lie in a caret, then the last two leaves of S do not lie in a caret and ii) for every integer k with $0 \leq k < n$, if $a_k > 0$ and $b_k > 0$ then either $a_{k+1} > 0$ or $b_{k+1} > 0$.*

Proof. To prove the first statement of the theorem, by composing functions it suffices to prove that the function with tree diagram $(R, \mathcal{T}_n)$ is $X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$.

The proof of this will proceed by induction on $a = \sum_{i=0}^{n} a_i$. If $a = 0$, then $R = \mathcal{T}_n$, and the result is clear. Now suppose that $a > 0$ and that the result is true for smaller values of a. Let m be the smallest index such that $a_m > 0$. Then there are ordered rooted binary subtrees R_1, R_2, R_3 of R such that R has the form of the tree at the left of Figure 10.

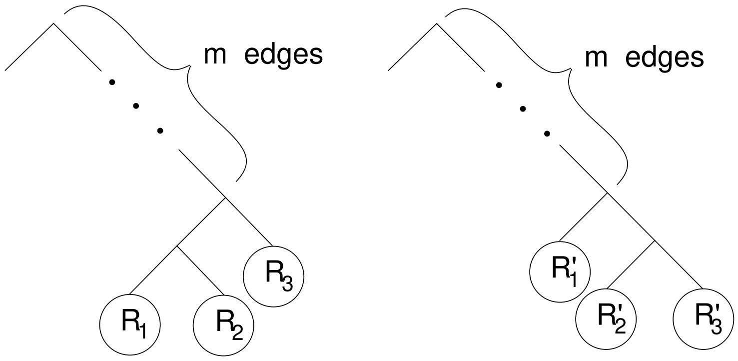

Figure 10. The $\mathcal{T}$-trees R and R'.

Let R' be the $\mathcal{T}$-tree shown at the right of Figure 10, where R_1', R_2', R_3' are isomorphic with R_1, R_2, R_3 as ordered rooted binary trees. According to Example 2.3, the function with tree diagram (R, R') is X_m^{-1}. If $a_0', \ldots, a_n'$ are the exponents of R', then $a_m' = a_m - 1$ and $a_k' = a_k$ if $k \neq m$. Thus the induction hypothesis applies to R', and so the function with tree diagram $(R', \mathcal{T}_n)$ is $X_n^{-a_n'} \cdots X_2^{-a_2'} X_1^{-a_1'} X_0^{-a_0'}$.

The second statement of the theorem is now easy to prove.

This proves Theorem 2.5. ∎

Corollary 2.6. *Thompson's group F is generated by A and B.*

Corollary-Definition 2.7. *Every nontrivial element of F can be expressed in unique normal form*

$$X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0},$$

where $n, a_0, \ldots, a_n, b_0, \ldots, b_n$ are nonnegative integers such that i) exactly one of a_n and b_n is nonzero and ii) if $a_k > 0$ and $b_k > 0$ for some integer k with $0 \leq k < n$, then $a_{k+1} > 0$ or $b_{k+1} > 0$. Furthermore, every such normal form function in F is nontrivial.

The functions in F of the form $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n}$ with $b_k \geq 0$ for $k = 0, \ldots, n$ will be called *positive*. The positive elements of F are exactly those with tree diagrams having domain tree $\mathcal{T}_n$ for some nonnegative integer n. Inverses of positive elements will be called *negative*.

Lemma 2.8. *The set of positive elements of F is closed under multiplication.*

Proof. Let f and g be positive elements of F. Let $(\mathcal{T}_m, R)$, respectively $(\mathcal{T}_n, S)$, be tree diagrams for f, respectively g. If the right side of S has length k, then it is easy to see that fg has a tree diagram with domain tree $\mathcal{T}_{n+\max\{m-k,0\}}$. Thus fg is positive. This proves Lemma 2.8. ∎

§3. Presentations for F.

Two presentations for F will be given in this section.

Now two groups F_1 and F_2 will be defined by generators and relations. The generators $A, B, X_0, X_1, X_2, \ldots$ will be referred to as *formal symbols*, as opposed to the functions defined above. Given elements x, y in a group, $[x, y] = xyx^{-1}y^{-1}$.

$$F_1 = \langle A, B : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$

$$F_2 = \langle X_0, X_1, X_2, \ldots : X_k^{-1} X_n X_k = X_{n+1} \quad \text{for} \quad k < n \rangle$$

Theorem 3.1. *There exists a group isomorphism from F_1 to F_2 which maps A to X_0 and B to X_1.*

Proof. There is a group homomorphism from the free group generated by the formal symbols A and B to F_2 such that A maps to X_0 and B maps to X_1. This homomorphism is surjective because $X_n = X_0^{-(n-1)}X_1X_0^{n-1}$ for $n \geq 2$. To see that the defining relations of F_1 are in the kernel of this homomorphism, note that

$$X_1^{-1}X_2X_1 = X_0^{-1}X_2X_0 \qquad \text{and} \qquad X_1^{-1}X_3X_1 = X_0^{-1}X_3X_0,$$

hence

$$[X_0X_1^{-1}, X_2] = 1 \qquad \text{and} \qquad [X_0X_1^{-1}, X_3] = 1,$$

hence

$$[X_0X_1^{-1}, X_0^{-1}X_1X_0] = 1 \qquad \text{and} \qquad [X_0X_1^{-1}, X_0^{-2}X_1X_0^2] = 1.$$

Thus to complete the proof of Theorem 3.1 it suffices to prove that there exists a group homomorphism from F_2 to F_1 which maps X_0 to A and X_1 to B. To prove this it in turn suffices, after setting $Y_0 = A$ and $Y_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$, to prove that

$$Y_k^{-1}Y_nY_k = Y_{n+1} \qquad \text{for } k < n. \tag{3.2}$$

A closely related statement is that

$$[A^{-1}B, Y_m] = 1 \qquad \text{for } m \geq 3. \tag{3.3}$$

Lines (3.2) and (3.3) will be proved in this paragraph. To see that line (3.3) is true for $m = 3$ note that

$$\begin{aligned}[AB^{-1}, A^{-1}BA] = 1 &\implies A^{-1}[AB^{-1}, A^{-1}BA]A = 1 \implies [B^{-1}A, A^{-2}BA^2] = 1 \\ &\implies [A^{-1}B, A^{-2}BA^2] = 1 \implies [A^{-1}B, Y_3] = 1.\end{aligned}$$

The same argument gives line (3.3) for $m = 4$. The following equations show that line (3.2) is true if line (3.3) is true for $m = n - k + 2$.

$$\begin{aligned}Y_nY_k &= A^{-n+1}BA^{n-1}A^{-k+1}BA^{k-1} = A^{-k+2}A^{-(n-k+1)}BA^{n-k+1}A^{-1}BA^{k-1} \\ &= A^{-k+2}Y_{n-k+2}A^{-1}BA^{k-1} = A^{-k+2}A^{-1}BY_{n-k+2}A^{k-1} \\ &= A^{-k+1}BA^{k-1}A^{-k+1}Y_{n-k+2}A^{k-1} = Y_kY_{n+1}\end{aligned}$$

Thus line (3.2) is true for every positive integer n and $k = n - 1$. In particular, $Y_3^{-1}Y_4Y_3 = Y_5$. Because line (3.3) is true for $m = 3$ and $m = 4$, it follows that line (3.3) is true for $m = 5$. An obvious induction argument now gives line (3.3) for every $m \geq 3$. This proves lines (3.2) and (3.3).

The proof of Theorem 3.1 is now complete. ∎

Theorem 3.4. *There exist group isomorphisms from F_1 and F_2 to F which map the formal symbols $A, B, X_0, X_1, X_2, \ldots$ to the corresponding functions in F.*

Proof. Example 1.2 shows that the interior of the support of the function AB^{-1} in F is disjoint from the supports of the functions $A^{-1}BA$, $A^{-2}BA^2$ in F, and so the functions A, B in F satisfy the defining relations of F_1. Thus there exists a group homomorphism from F_1 to F which maps the formal symbols A, B to the corresponding functions in F. Corollary 2.6 shows that this group homomorphism is surjective. Theorem 3.1 shows that this surjective group homomorphism induces a surjective group homomorphism from F_2 to F which maps the formal symbols $X_0, X_1, X_2, \ldots$ to the corresponding functions in F. To prove Theorem 3.4 it suffices to prove that this latter group homomorphism is injective.

It will be proved that this latter group homomorphism is injective in this paragraph. The defining relations of F_2 imply that

$$X_k^{-1}X_n = X_{n+1}X_k^{-1}, \qquad X_n^{-1}X_k = X_kX_{n+1}^{-1}, \qquad X_nX_k = X_kX_{n+1} \qquad \text{for } k < n.$$

It follows that every nontrivial element x of F_2 can be expressed as a positive element times a negative element as in Corollary-Definition 2.7. If X_k occurs in both the positive and negative part of x but X_{k+1} occurs in neither, then because $X_kX_{n+1}X_k^{-1} = X_n$ for $n > k$, it is possible to simplify x by deleting one occurrence of X_k from both the positive and negative part of x and replacing every occurrence of X_{n+1} in x by X_n for $n > k$. Thus every nontrivial element of F_2 can be put in normal form as in Corollary-Definition 2.7. It follows from Corollary-Definition 2.7 that every nontrivial element of F_2 maps to a nontrivial element of F, as desired.

This proves Theorem 3.4. ∎

§4. Further properties of F.

As was realized by Thompson, whether F is amenable or not is an interesting question. Brin and Squier proved in Theorem 3.1 of [B-S] that F contains no free subgroups of rank greater than 1 (Corollary 4.9 here). Hence if F is not amenable, then F is a finitely-presented counterexample to von Neumann's conjecture: a discrete group is not amenable if and only if it contains a subgroup which is free of rank 2. See page 15 of [P] or see Conjecture 12.27 of [W]. We prove in Theorem 4.10 that F is not an elementary amenable group. Hence if F is amenable, then F is a finitely-presented counterexample to the conjecture that every discrete amenable group is an elementary amenable group. Both conjectures are false for finitely-generated groups; see [P] or [W] for details and references.

Theorem 4.1. *The commutator subgroup $[F,F]$ of F consists of all elements in F which are trivial in neighborhoods of 0 and 1. Furthermore, $F/[F,F] \cong \mathbf{Z} \oplus \mathbf{Z}$.*

Proof. There exists a group homomorphism $\varphi \colon F \to \mathbf{Z} \oplus \mathbf{Z}$ such that if $f \in F$, then $\varphi(f) = (a,b)$, where the right derivative of f at 0 is 2^a and the left derivative of f at 1 is 2^b. Since $\varphi(A) = (-1,1)$ and $\varphi(B) = (0,1)$, φ is surjective. It is easy to see that if K is a group generated by two elements and there exists a surjective group homomorphism from K to $\mathbf{Z} \oplus \mathbf{Z}$, then the kernel of that homomorphism is the commutator subgroup of K. Corollary 2.6 shows that F is generated by A and B, and so $[F,F] = Ker(\varphi)$. This proves Theorem 4.1. ∎

Lemma 4.2. *If $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ are partitions of $[0,1]$ consisting of dyadic rational numbers, then there exists $f \in F$ such that $f(x_i) = y_i$ for $i = 0, \dots, n$. Furthermore, if $x_{i-1} = y_{i-1}$ and $x_i = y_i$ for some i with $1 \le i \le n$, then f can be taken to be trivial on the interval $[x_{i-1}, x_i]$.*

Proof. Let m be a positive integer such that $2^m x_i \in \mathbf{Z}$ and $2^m y_i \in \mathbf{Z}$ for $i = 0, \dots, n$. Let $R = S$ be the $\mathcal{T}$-tree whose leaves consist of the standard dyadic intervals of length 2^{-m}. Let I be the leaf of R whose right endpoint is x_1, and let J be the leaf of S whose right endpoint is y_1. By adjoining carets to R with roots not right of I or adjoining carets to S with roots not right of J, it may be assumed that there are as many leaves in R left of I as there are in S left of J. Continue in this way to enlarge R and S if necessary so that the function f with tree diagram (R,S) maps x_i to y_i for $i = 0, \dots, n$. This easily proves Lemma 4.2. ∎

Theorem 4.3. *Every proper quotient group of F is Abelian.*

Proof. Let N be a nontrivial normal subgroup of F. It must be proved that F/N is Abelian.

For this it will be shown in this paragraph that the center of F is trivial. Let f be in the center of F. Since f commutes with B, f and f^{-1} stabilize the fixed point set of B, namely, $[0, \frac{1}{2}] \cup \{1\}$. This implies that $f(\frac{1}{2}) = \frac{1}{2}$. Because every element of F commutes with f, every element of F stabilizes the fixed point set of f. This and Lemma 4.2 easily imply that the fixed point set of f is $[0,1]$. Thus the center of F is trivial.

Because N contains a nontrivial element and the center of F is trivial, N contains a nontrivial commutator of F. Let $f = X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$ be such an element expressed in normal form. It is easy to see that then $a_0 = b_0$. Let k be the smallest index such that $a_k \neq b_k$. By replacing f by f^{-1} if necessary, it may be assumed that $b_k > a_k$. By replacing f by $X_k^{-a_k} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0} f X_0^{a_0} X_1^{a_1} X_2^{a_2} \cdots X_k^{a_k}$, it may be assumed that $b_0 = \cdots = b_{k-1} = 0$, $a_0 = \cdots = a_k = 0$ and $b_k > 0$. By replacing f by $X_0^{k-1} f X_0^{1-k}$ it may be assumed that $a_0 = a_1 = b_0 = 0$ and $b_1 > 0$. In this case $(X_0^{-1} f X_0)(X_1^{-1} f X_1)^{-1} = X_2^{b_1} X_1^{-b_1}$. Hence N contains $X_1^{-b}(X_2^b X_1^{-b}) X_1^b = X_1^{-b} X_2^b$ for some positive integer b. Hence N contains

$$\begin{aligned} X_0 X_2^{b-1}(X_2(X_1^{-b}X_2^b)X_2^{-1}(X_2^{-b}X_1^b))X_2^{1-b}X_0^{-1} &= X_0 X_2^{b-1}(X_2 X_{b+2}^{-1})X_2^{1-b}X_0^{-1} \\ &= X_0 X_2 X_3^{-1} X_0^{-1} = X_1 X_2^{-1} = BA^{-1}B^{-1}A. \end{aligned}$$

Thus F/N is Abelian.

This proves Theorem 4.3. ∎

Lemma 4.4. *Let a, b be dyadic rational numbers with $0 \leq a < b \leq 1$ such that $b-a$ is a power of 2. Then the subgroup of F consisting of all functions with support in $[a,b]$ is isomorphic with F by means of the straightforward linear conjugation.*

Proof. Let $\varphi\colon [a,b] \to [0,1]$ be the linear homeomorphism defined by $\varphi(x) = \frac{1}{b-a}x - \frac{a}{b-a}$. Then $\varphi^{-1}\colon [0,1] \to [a,b]$ is given by $\varphi^{-1}(x) = (b-a)x + a$. The isomorphism from F to the subgroup in question is defined so that for every $f \in F$, $f \mapsto \varphi^{-1} f \varphi$. The derivative of $\varphi^{-1} f \varphi$ is $f'\varphi$. The functions φ and φ^{-1} both map dyadic rational numbers to dyadic rational numbers. Thus f is a function from [0,1] to [0,1] whose points of nondifferentiability are dyadic rational numbers if and only if $\varphi^{-1} f \varphi$ is a function from $[a,b]$ to $[a,b]$ whose points of nondifferentiability are dyadic rational numbers. Lemma 4.4 easily follows. ∎

Theorem 4.5. *The commutator subgroup $[F,F]$ of F is a simple group.*

Proof. Let N be a normal subgroup of $[F,F]$ containing a nontrivial element f. According to Theorem 4.1, f is trivial in a neighborhood of 0 and a neighborhood of 1. Theorem 4.1 and Lemma 4.2 easily imply that there exists $g \in [F,F]$ which maps neighborhoods of the intervals $[0,\frac{1}{4}]$ and $[\frac{3}{4},1]$ into these neighborhoods of 0 and 1. Thus $g^{-1}fg$ is a nontrivial function in N whose support lies in $[\frac{1}{4},\frac{3}{4}]$. According to Lemma 4.4 the subgroup of all functions in F with support in $[\frac{1}{4},\frac{3}{4}]$ is isomorphic with F. Now Theorem 4.3 shows that N contains the commutator subgroup of the subgroup of F of all functions with support in $[\frac{1}{4},\frac{3}{4}]$. Thus N contains all functions in F which are trivial in neighborhoods of the intervals $[0,\frac{1}{4}]$ and $[\frac{3}{4},1]$. Just as the above function f is conjugated by g into this set of functions, every element of $[F,F]$ is $[F,F]$-conjugate to a function in this set.

This proves Theorem 4.5. ∎

Theorem 4.6. *The submonoid of F generated by A, B, B^{-1} is the free product of the submonoid generated by A and the subgroup generated by B.*

Proof. The proof will deal with reduced words in A, B, B^{-1}. Given such a reduced word w, let $\overline{w}$ denote the corresponding element in F. What must be shown is that if w_1 and w_2 are two reduced words in A, B, B^{-1} with $\overline{w_1} = \overline{w_2}$, then $w_1 = w_2$.

Suppose that there exist reduced words w_1, w_2 in A, B, B^{-1} with $\overline{w_1} = \overline{w_2}$ and $w_1 \neq w_2$. Choose such words w_1 and w_2 so that the sum of their lengths is minimal. Suppose that one of w_1 and w_2 ends (on the right) with B and the other ends with B^{-1}. Then $\overline{w_1 B} = \overline{w_2 B}$, $w_1 B \neq w_2 B$ and the sum of the lengths of $w_1 B$ and $w_2 B$ is minimal. Thus by multiplying w_1 and w_2 on the right by an appropriate power of B, it may further be assumed that w_1 ends with A. Because the sum of the lengths of w_1 and w_2 is minimal, w_2 ends with either B or B^{-1}.

There exists a group homomorphism $\varphi\colon F \to \mathbf{Z}$ such that $\varphi(A) = 1$ and $\varphi(B) = 0$. Hence $\varphi(\overline{w_1}) = \varphi(\overline{w_2})$ implies that the number of A's which occur in w_1 equals the number of A's which occur in w_2. Let n be this number of A's. Clearly $n > 0$.

Now note that $A(\frac{3}{4}) = \frac{1}{2}$ and moreover $A^n(\frac{3}{4}) = 2^{-n}$. Because B and B^{-1} act trivially on the closed interval $[0,\frac{1}{2}]$, it follows that $\overline{w_1}(\frac{3}{4}) = 2^{-n}$.

Suppose that w_2 ends with B. Then w_2 ends with AB^m for some positive integer m. Note that $\frac{1}{2} < B^m(\frac{3}{4}) < \frac{3}{4}$, and so $\frac{1}{4} < AB^m(\frac{3}{4}) < \frac{1}{2}$. Again because B and B^{-1} act trivially on the closed interval $[0,\frac{1}{2}]$, it follows that $\overline{w_2}(\frac{3}{4})$ is not a power of 2, contrary to the fact that $\overline{w_1}(\frac{3}{4}) = 2^{-n}$.

Thus w_2 ends with B^{-1}. Now note that $\frac{7}{8} \leq B^{-1}(x) = A^{-1}(x)$ for every x in the interval $[\frac{3}{4},1]$. Because $\overline{w_2}(\frac{3}{4}) = 2^{-n}$, it follows that $w_2 = w_3 w_4$, where w_3 and w_4 are reduced words in A, B, B^{-1} with $w_4(\frac{3}{4}) = \frac{3}{4}$ and w_3 ends with either A or B. If w_3 ends with A, then the argument of the penultimate paragraph shows that $\overline{w_2}(\frac{3}{4}) = \overline{w_3}(\frac{3}{4}) = 2^{-n'}$, where n' is the number of A's in w_3. But A occurs in w_4 because $w_4(\frac{3}{4}) = \frac{3}{4}$, and so $n' < n$. This is impossible, and so w_3 ends with B. The argument of the previous paragraph shows in this case that $\overline{w_2}(\frac{3}{4})$ is not even a power of 2. This contradiction completes the proof of the theorem. ∎

Corollary 4.7. *Thompson's group F has exponential growth.*

Theorem 4.8. *Every non-Abelian subgroup of F contains a free Abelian subgroup of infinite rank.*

Proof. Let K be a subgroup of F generated by elements f, g such that $[f,g] \neq 1$. Let $I_1, \ldots, I_n$ be the closed intervals in $[0,1]$ with nonempty interiors such that for every integer k with $1 \leq k \leq n$, if x is an endpoint of I_k, then $f(x) = g(x) = x$ and if x is an interior point of I_k, then either $f(x) \neq x$ or $g(x) \neq x$.

In this paragraph it will be shown for every integer k with $1 \le k \le n$ that the endpoints of I_k are cluster points of the K-orbit of every interior point of I_k. Let x be an interior point of I_k. Let y be the greatest lower bound of the K-orbit of x. If y is not the left endpoint of I_k, then either $f(y) \neq y$ or $g(y) \neq y$. Suppose that $f(y) \neq y$. Then either $f(y) < y$ or $f^{-1}(y) < y$. Hence there exists a neighborhood of y such that every element of its image under either f or f^{-1} is less than y. Thus y is the left endpoint of I_k. The same argument applies to least upper bounds. This proves for every integer k with $1 \le k \le n$ that the endpoints of I_k are cluster points of the K-orbit of every interior point of I_k.

Let $h_1 = [f,g]$. Just as commutators in F are trivial in neighborhoods of 0 and 1, h_1 is trivial in neighborhoods of the endpoints of I_1. The result of the previous paragraph implies that h_1 is conjugate in K to a function h_2 whose support in I_1 is disjoint from the support of h_1 in I_1. It easily follows that there exists an infinite sequence of functions $h_1, h_2, h_3, \dots$ in K whose supports in I_1 are mutually disjoint. Thus $[h_i, h_j]$ is trivial on I_1 for all positive integers i, j. If $[h_i, h_j] = 1$ for all positive integers i and j, then it is easy to see that $h_1, h_2, h_3, \dots$ form a basis of a free Abelian subgroup of K, as desired.

If $[h_i, h_j] \neq 1$ for some positive integers i and j, then repeat the argument of the previous paragraph with h_1 replaced by this nontrivial commutator $[h_i, h_j]$ and I_1 replaced by some interval I_k on which $[h_i, h_j]$ is not trivial. This process eventually leads to an infinite sequence of functions $h_1, h_2, h_3, \dots$ in K which form a basis of a free Abelian subgroup.

This proves Theorem 4.8. ∎

Corollary 4.9. *Thompson's group F does not contain a non-Abelian free group.*

The next result relies on the paper [C] by Ching Chou. Elementary amenable groups are the groups which can be gotten from finite groups and Abelian groups by repeatedly taking: 1) subgroups, 2) factor groups, 3) group extensions and 4) direct unions.

Theorem 4.10. *Thompson's group F is not an elementary amenable group.*

Proof. According to (a) of Chou's Proposition 2.2, it suffices to prove that $F \notin EG_\alpha$ for every ordinal α. Since EG_0 consists of finite groups and Abelian groups, it is clear that $F \notin EG_0$, so assume that $\alpha > 0$ and that $F \notin EG_\beta$ for every ordinal $\beta < \alpha$.

If α is a limit ordinal, then there is nothing to prove. Suppose that α is not a limit ordinal. It must be shown that F cannot be constructed from groups in $EG_{\alpha-1}$ as a group extension or as a direct union.

First consider group extensions. Suppose that F contains a normal subgroup N such that N, $F/N \in EG_{\alpha-1}$. Since $F \notin EG_{\alpha-1}$, N is nontrivial. Theorem 4.3 implies that $[F,F] \subset N$. Now Theorem 4.1 and Lemma 4.4 easily imply that N contains a subgroup isomorphic with F. Proposition 2.1 of [C] states that subgroups of groups in $EG_{\alpha-1}$ are also in $EG_{\alpha-1}$. Thus $F \in EG_{\alpha-1}$, contrary to hypothesis. This proves that F cannot be constructed from $EG_{\alpha-1}$ as a group extension.

Second consider direct unions. Suppose that F is a direct union of groups in $EG_{\alpha-1}$. This is clearly impossible because F is finitely generated.

This proves Theorem 4.10. ∎

We conclude Section 4 by showing that F is a totally ordered group. Define the set of *order positive* elements of F to be the set P of functions $f \in F$ such that there exists a subinterval $[a,b]$ of $[0,1]$ on which the derivative of f is less than 1 and $f(x) = x$ for $0 \le x \le a$. It is easy to see that the positive elements of F are indeed order positive. It is clear that $F = P^{-1} \cup \{1\} \cup P$. It is easy to see that P is closed under multiplication and $f^{-1}Pf \subset P$ for every $f \in F$. This proves Theorem 4.11.

Theorem 4.11. *Thompson's group F is a totally ordered group.*

§5. Thompson's group T.

The material in this section is mainly from unpublished notes of Thompson.

Consider S^1 as the interval $[0,1]$ with the endpoints identified. Then T is the group of orientation-preserving piecewise linear homeomorphisms from S^1 to itself that map images of dyadic rational numbers to images of dyadic rational numbers and that are differentiable except at finitely many images of dyadic rational numbers and on intervals of differentiability the derivatives are powers of 2. Just as we proved that F is a group, it is easy to see that T is indeed a group.

Example 5.1. The elements A and B of F induce elements of T, which will still be denoted by A and B. A third element of T is the function C defined (on $[0,1]$) by

$$C(x) = \begin{cases} \frac{x}{2}+\frac{3}{4}, & 0 \le x \le \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x \le \frac{3}{4} \\ x-\frac{1}{4}, & \frac{3}{4} \le x \le 1. \end{cases}$$

We can associate tree diagrams and unique reduced tree diagrams to elements of T almost exactly as we did to elements of F. The only difference is the following. Elements of F map leftmost leaves of domain trees to leftmost leaves of range trees. When an element of T does not do this, we denote the image in its range tree of the leftmost leaf of its domain tree with a small circle. For example, the reduced tree diagram for C is in Figure 11.

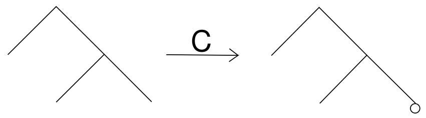

Figure 11. The reduced tree diagram for C.

Lemma 5.2. *The elements A, B and C generate T and satisfy the following relations:*

1) $[AB^{-1}, A^{-1}BA] = 1$;
2) $[AB^{-1}, A^{-2}BA^2] = 1$;
3) $C = B(A^{-1}CB)$;
4) $(A^{-1}CB)(A^{-1}BA) = B(A^{-2}CB^2)$;
5) $CA = (A^{-1}CB)^2$; and
6) $C^3 = 1$.

Proof. Let H be the subgroup of T generated by $\{A, B, C\}$. Since $\{A, B\}$ is a generating set for F, $F \subset H$. Suppose $f \in T$. Let $[x] = f([0])$. If $[x] = [0]$, then $f \in F$ and hence $f \in H$. If $[x] \ne [0]$, then there is an element $h \in F$ with $h(x) = \frac{3}{4}$ by Lemma 4.2. Then $g = C^{-1}hf$ fixes $[0]$, so $g \in F$. Hence $f = h^{-1}Cg \in H$ and $H = T$. Thus A, B and C generate T.

Relations 1) and 2) are proved in Section 3.

Consider relation 3). It is equivalent to the relation $CBC^{-1} = AB^{-1}$. The reduced tree diagram for CBC^{-1} is computed in Figure 12, the notation being straightforward.

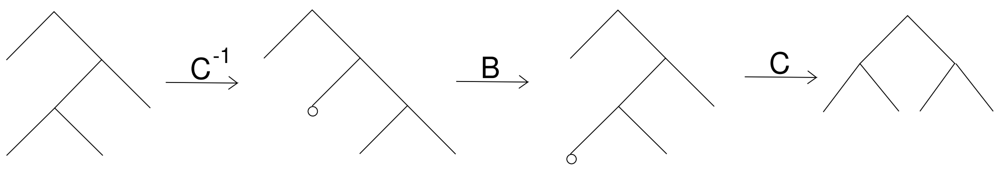

Figure 12. Computing the reduced tree diagram for CBC^{-1}.

Referring to Figure 1 shows that AB^{-1} has the same reduced tree diagram as CBC^{-1}, which completes the verification of relation 3).

Consider relation 4). It is equivalent to

$$(B^{-1}C^{-1}A)(AB^{-1})(A^{-1}CB) = BA^{-1}B^{-1}A,$$

where the term $A^{-1}CB$ here corresponds to the same term in relation 4). We compute a tree diagram for the left side of this equation in Figure 13.

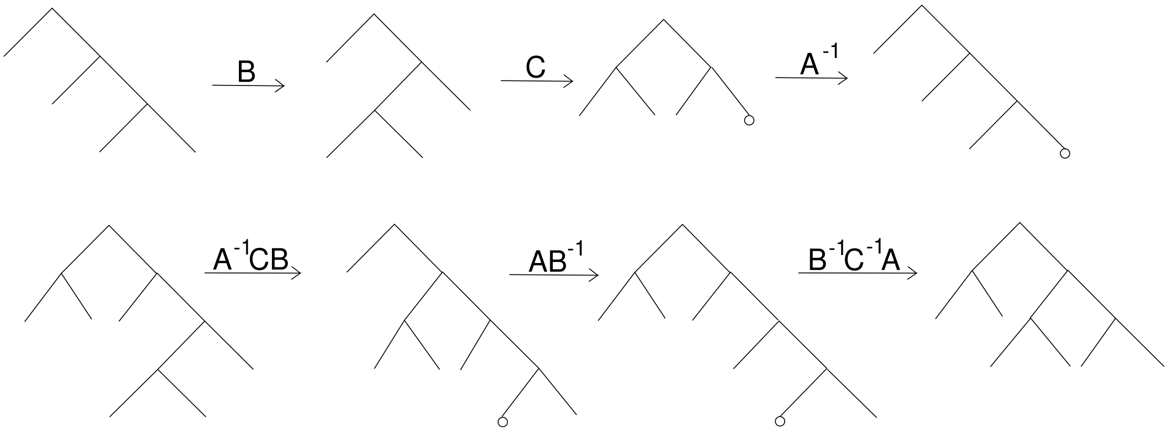

Figure 13. Computing a tree diagram for $(B^{-1}C^{-1}A)(AB^{-1})(A^{-1}CB)$.

Referring to Figure 1 now completes the verification of relation 4).

Relation 6) is easily verified using the reduced tree diagram for C.

Finally consider relation 5). Use relation 6) and then relation 3) to rewrite relation 5): $CA = A^{-1}CBA^{-1}CB \Leftrightarrow CA = A^{-1}C^{-1}(C^{-1}BA^{-1}CB) \Leftrightarrow CA = A^{-1}C^{-1} \Leftrightarrow (AC)^2 = 1$. The reduced tree diagram for AC is computed in Figure 14.

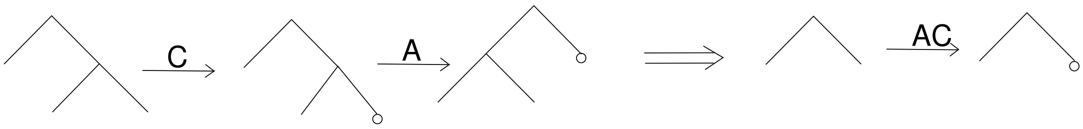

Figure 14. Computing the reduced tree diagram for AC.

Hence AC acts on S^1 by translation by $[\frac{1}{2}]$, and so $(AC)^2 = 1$, which gives relation 5). ∎

Let

$$T_1 = \langle A, B, C : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2], C^{-1}B(A^{-1}CB), \\ ((A^{-1}CB)(A^{-1}BA))^{-1}B(A^{-2}CB^2), (CA)^{-1}(A^{-1}CB)^2, C^3\rangle.$$

Lemma 5.3. *There is a surjection $T_1 \to T$ that maps the formal symbols A, B, and C to the functions A, B, and C in T.*

Proof. This follows immediately since the functions A, B, and C satisfy the relations 1) - 6). ∎

Lemma 5.4. *The subgroup of T_1 generated by A and B is isomorphic to F.*

Proof. The results of Section 3 show that there exists a group homomorphism from F to the subgroup of T_1 generated by A and B whose composition with the map from T_1 to T is the identity map on F. This proves Lemma 5.4. ∎

It is easier at this point to prove that T is simple than to prove that T_1 is simple. However, it is preferable to prove that T_1 is simple, since then Lemma 5.3 implies that T is isomorphic to T_1.

Define the elements X_n, $n \geq 0$, of T_1 by $X_0 = A$ and $X_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$. It follows from Theorem 3.4 and Lemma 5.4 that $X_nX_k = X_kX_{n+1}$ if $k < n$. Define the elements Y_n, $n \geq 1$, of T_1 by $Y_n = A^{-(n-1)}CB^{n-1}$. For convenience we define $Y_0 = 1$.

Lemma 5.5. *If k, n are positive integers and $k \leq n$, then*
i) $Y_n = X_nY_{n+1}$;
ii) $Y_nX_k = X_{k-1}Y_{n+1}$; and
iii) $Y_nA = Y_{n+1}^2$.

Proof.

$$Y_n = A^{-(n-1)}CB^{n-1} = A^{-(n-1)}B(A^{-1}CB)B^{n-1} = (A^{-(n-1)}BA^{n-1})(A^{-n}CB^n) = X_nY_{n+1},$$

which proves i).

If $k = 1$, ii) follows from the definition. If $k = n = 2$, ii) follows from relation 4). If $k = 2$ and $n > 2$, then by induction on n

$$Y_n X_2 \overset{i)}{=} X_{n-1}^{-1} Y_{n-1} X_2 \overset{\text{induct}}{=} X_{n-1}^{-1} X_1 Y_n = X_1 X_n^{-1} Y_n \overset{i)}{=} X_1 Y_{n+1}.$$

If $k \geq 3$, then by induction on k

$$Y_n X_k = A^{-1} Y_{n-1} B X_k = A^{-1} Y_{n-1} X_{k-1} B \overset{\text{induct}}{=} A^{-1} X_{k-2} Y_n B = (A^{-1} X_{k-2} A)(A^{-1} Y_n B) = X_{k-1} Y_{n+1}.$$

Equation iii) follows by induction on n. If $n = 1$ then it is relation 5). If $n > 1$, then

$$Y_n A = A^{-1} Y_{n-1} B A = A^{-1} Y_{n-1} A X_2 \overset{\text{induct}}{=} A^{-1} Y_n^2 X_2 \overset{ii)}{=} A^{-1} Y_n B Y_{n+1} = Y_{n+1}^2. \blacksquare$$

Lemma 5.6. *If n is a positive integer, $m \in \{1, \ldots, n+1\}$, and $r, s \in \{0, \ldots, n\}$, then*

i)

$$Y_n^m X_r = \begin{cases} X_{r-m} Y_{n+1}^m, & r \geq m \\ Y_{n+1}^{m+1}, & r = m-1 \\ X_{r+(n+2-m)} Y_{n+1}^{m+1}, & r < m-1; \end{cases}$$

ii)

$$X_s^{-1} Y_n^m = \begin{cases} Y_{n+1}^{m+1} X_{(s+m)-(n+2)}^{-1}, & s \geq (n+2)-m \\ Y_{n+1}^m, & s = (n+1)-m \\ Y_{n+1}^m X_{s+m}^{-1}, & s \leq n-m; \end{cases}$$

iii) $Y_n^m = X_{(n+1)-m} Y_{n+1}^m$;

iv) $Y_n^m = Y_{n+1}^{m+1} X_{m-1}^{-1}$;

v) $Y_n^{n+2} = 1$.

Proof. The first line of i) follows from 5.5.ii). If $r = 0$, the second line is 5.5.iii); if $r > 0$

$$Y_n^m X_r = Y_n Y_n^r X_r \overset{5.5.ii)}{=} Y_n A Y_{n+1}^r \overset{5.5.iii)}{=} Y_{n+1}^2 Y_{n+1}^r = Y_{n+1}^{m+1}.$$

This proves i) if $r \geq m-1$.

$$Y_n^m = Y_n^{m-1} Y_n \overset{5.5.i)}{=} Y_n^{m-1} X_n Y_{n+1} \overset{5.5.ii)}{=} X_{n-(m-1)} Y_{n+1}^m,$$

which proves iii). If $r < m-1$, then

$$Y_n^m X_r = Y_n^{m-(r+1)} Y_n^{r+1} X_r \overset{5.5}{=} Y_n^{m-(r+1)} Y_{n+1}^{r+2} \overset{iii)}{=} X_{n+1-(m-(r+1))} Y_{n+1}^{m-(r+1)} Y_{n+1}^{r+2} = X_{r+(n+2-m)} Y_{n+1}^{m+1},$$

which finishes the proof for i).

The first line of ii) follows from i), with $s = r + (n+2-m)$. The second line of ii) follows from iii). The third line of ii) follows from i), with $s = r - m$. Equation iv) follows from the second line of i). If $t \geq 1$, then

$$Y_t^{t+2} = Y_t Y_t^{t+1} \overset{iii)}{=} Y_t A Y_{t+1}^{t+1} \overset{5.5.iii)}{=} Y_{t+1}^2 Y_{t+1}^{t+1} = Y_{t+1}^{t+3}.$$

Since $Y_1^3 = C^3 = 1$, this proves v) and completes the proof of Lemma 5.6. $\blacksquare$

Following the terminology for F, an element of T_1 which is a product of nonnegative powers of the X_i's will be called *positive* and an inverse of a positive element will be called *negative*.

Theorem 5.7. *If $g \in T_1$, then $g = pY_n^m q^{-1}$ for some positive elements p, q and nonnegative integers m, n with $m < n+2$.*

Proof. We first show that if i, j, k and l are positive integers, then there are positive elements p and q and nonnegative integers m and n such that $Y_j^i Y_l^k = pY_n^m q^{-1}$. Suppose that i, j, k and l are positive integers and that $g = Y_j^i Y_l^k$. Since $Y_j^{j+2} = Y_l^{l+2} = 1$ by Lemma 5.6.v), we can assume that $i < j+2$ and $k < l+2$. Let $n \geq \max\{j, l\}$. By Lemma 5.6.iii) and Lemma 5.6.iv), there is a positive integer r and there are positive elements p and q such that $Y_j^i = pY_n^i$ and $Y_l^k = Y_n^r q^{-1}$. Hence $Y_j^i Y_l^k = pY_n^{i+r} q^{-1}$.

Let $H = \{g \in T_1 : g = pY_n^m q^{-1}$ for some positive elements p, q and nonnegative integers m, n with $m < n+2\}$. Lemma 5.6.v) easily implies that H is closed under inversion. To show that H is closed under multiplication, suppose that $g_1, g_2 \in H$. Then $g_1 = p_1 Y_j^i q_1^{-1}$ and $g_2 = p_2 Y_l^k q_2^{-1}$ for some positive elements p_1, p_2, q_1 and q_2 and some nonnegative integers i, j, k and l with $i < j+2$ and $k < l+2$. By Corollary 2.7, there are positive elements p_3 and q_3 such that $q_1^{-1} p_2 = p_3 q_3^{-1}$. Hence $g_1 g_2 = p_1 Y_j^i q_1^{-1} p_2 Y_l^k q_2^{-1} = p_1 Y_j^i p_3 q_3^{-1} Y_l^k q_2^{-1}$. Lemma 5.6.iii), Lemma 5.6.iv) and Lemma 2.8, which states that the set of positive elements of F is closed under multiplication, show that we may assume that if $j > 0$ and X_r occurs in p_3, then $j \geq r$ and if $l > 0$ and X_s occurs in q_3, then $l \geq s$. Now Lemma 5.6.i), Lemma 5.6.ii) and Lemma 2.8 show that there are positive elements p_4 and q_4 and nonnegative integers r, s, t and u such that $g_1 g_2 = p_4 Y_s^r Y_u^t q_4^{-1}$. By the previous paragraph and Lemma 2.8, there are positive elements p_5 and q_5 and nonnegative integers m and n such that $g_1 g_2 = p_5 Y_n^m q_5^{-1}$. Since we can assume that $m < n+2$ by Lemma 5.6.v), $g_1 g_2 \in H$. Hence H is a subgroup of T_1. Since T_1 is generated by $A = X_0$, $B = X_1$ and $C = Y_1$, all of which are in H, $H = T_1$. ∎

Theorem 5.8. *T_1 is simple.*

Proof. Suppose N is a nontrivial normal subgroup of T_1, and let $\theta\colon T_1 \to T_1/N$ be the quotient homomorphism. Then there is an element $g \in T_1$ with $g \neq 1$ and $\theta(g) = 1$. By Theorem 5.7, $g = pY_n^m q^{-1}$ for some positive elements p, q and nonnegative integers m, n with $m < n+2$. Then $\theta(p^{-1}q) = \theta(Y_n^m)$ and $\theta((p^{-1}q)^{n+2}) = \theta((Y_n^m)^{n+2}) = \theta((Y_n^{n+2})^m) \overset{5.6.v)}{=} \theta(1) = 1$. By Lemma 5.4, there is a homomorphism $\alpha\colon F \to T_1/N$ defined on generators by $\alpha(A) = \theta(A)$ and $\alpha(B) = \theta(B)$. If $p^{-1}q \neq 1$, then $(p^{-1}q)^{n+2} \neq 1$, and so $\alpha(F)$ is a proper quotient group of F. Since every proper quotient group of F is Abelian by Theorem 4.3, $\theta(AB) = \theta(BA)$. If $p^{-1}q = 1$, then $m, n > 0$ and $1 = \theta(Y_n^m) \overset{5.6.iv)}{=} \theta(Y_{n+1}^{m+1})\theta(X_{m-1}^{-1})$ and hence $\theta(X_{m-1}^{n+3}) = \theta((Y_{n+1}^{m+1})^{n+3}) = \theta((Y_{n+1}^{n+3})^{m+1}) = \theta(1) = 1$. It follows as before that $\theta(AB) = \theta(BA)$. Hence $\theta(A^{-1}BA) = \theta(B)$, so by 4) $\theta(A^{-1}C) = \theta(BA^{-2}C)$. Hence $\theta(BA^{-1}) = 1$, and by 3) $\theta(B) = 1$. This implies that $\theta(A) = 1$. It now follows from 5) that $\theta(C) = 1$. Thus $N = T_1$, and so T_1 is simple. ∎

Corollary 5.9. *T_1 is isomorphic to T.*

§6. Piecewise integral projective structures.

The definition of piecewise integral projective structures is due to W. Thurston. These structures arise naturally as structures on the boundaries of the Teichmüller spaces of surfaces. The interpretations of F and T as $PIP^+(\Delta_1)$ and $PIP^+(S^1)$ are also due to Thurston; we learned this from him in 1975.

Fix a positive integer n.

The symbol Δ_n denotes the n-simplex $\{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1$ and $x_i \geq 0$ for all $i\}$. The n-simplex Δ_n is an orientable n-manifold with boundary. A *rational point* of Δ_n is a point $(x_1, \ldots, x_{n+1}) \in \Delta_n$ with each $x_i \in \mathbf{Q}$.

Set $\mathbf{R}_+^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : x_i \geq 0$ for $i = 1, \ldots, n+1\}$. One defines $\rho : \mathbf{R}_+^{n+1} \setminus \{0\} \to \Delta_n$ using the projective structure of $\mathbf{R}^{n+1}$; that is, $\rho(x) = \frac{x}{|x|}$, where $|x| = \sum_{i=1}^{n+1} |x_i|$. Let $U \subset \Delta_n$. A function $f\colon U \to \Delta_n$ is *integral projective* if there exists $A \in GL(n+1, \mathbf{Z})$ such that $U \subset \{x \in \Delta_n : A(x) \in \mathbf{R}_+^{n+1}\}$ and $f = \rho \circ A\big|_U$. It is easily seen that an integral projective map is a homeomorphism onto its image.

A *rational subsimplex* of Δ_n is a subsimplex of Δ_n in which each vertex is a rational point; a *rational subdivision* of Δ_n is a simplicial subdivision in which each n-simplex is a rational subsimplex. An *integral subsimplex* of Δ_n is a subsimplex of Δ_n which is homeomorphic to Δ_n by an integral projective map. Similarly, an *integral subdivision* of Δ_n is a simplicial subdivision of Δ_n in which each n-simplex is an integral subsimplex of Δ_n.

A *piecewise integral projective (PIP)* homeomorphism of Δ_n is a homeomorphism $f\colon \Delta_n \to \Delta_n$ such that there is an integral subdivision $\mathcal{S}$ of Δ_n with $f\big|_\sigma$ integral projective for each simplex σ of $\mathcal{S}$. Define $PIP(\Delta_n)$ to be the set of all PIP homeomorphisms of Δ_n. We wish to prove that $PIP(\Delta_n)$ is a group by proving that it is closed under inversion and composition. It is easy to see that $PIP(\Delta_n)$ is closed under inversion. It is not immediately obvious that the composition of two PIP homeomorphisms is a PIP homeomorphism. The stumbling block is whether two integral subdivisions of Δ_n have a common refinement which is an integral subdivision. According to Exercise 5 on page 15 of [R-S] their intersection is a cell complex which is a common refinement of both, and it is easy to see that the cells of this intersection complex have rational points as vertices. Proposition 2.9 of [R-S] states that such a cell complex can be subdivided to a simplicial complex without introducing any new vertices. Hence to prove that $PIP(\Delta_n)$ is a group it suffices to prove the following theorem.

Theorem 6.1. *Every rational subdivision of Δ_n has a refinement that is an integral subdivision.*

Proof. We define the *lift* of a rational point x in Δ_n to be the unique point $\widetilde{x}$ in $\mathbf{Z}^{n+1} \cap \mathbf{R}_+^{n+1}$ such that $\rho(\widetilde{x}) = x$ and the greatest common divisor of the coordinates of $\widetilde{x}$ is 1. We define the *index* of an n-dimensional rational subsimplex σ of Δ_n as follows. Let $v_1, \ldots, v_{n+1}$ be the vertices of σ. Then the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v_1}, \ldots, \widetilde{v_{n+1}}$ has finite index in $\mathbf{Z}^{n+1}$. The index $\operatorname{ind}(\sigma)$ of σ is by definition this index. Equivalently, $\operatorname{ind}(\sigma) = |\det(v_1, \ldots, v_{n+1})|$, the absolute value of the determinant of the matrix whose columns are $v_1, \ldots, v_{n+1}$. It is easy to see that $\operatorname{ind}(\sigma) = 1$ if and only if σ is integral.

The argument will proceed as follows. Let $\mathcal{S}$ be a rational subdivision of Δ_n. Suppose that σ is an n-simplex in $\mathcal{S}$ with $\operatorname{ind}(\sigma) > 1$. A rational point v in σ will be suitably chosen. We will let $\mathcal{R}$ be the simplicial complex obtained from $\mathcal{S}$ by starring at v as on page 15 of [R-S]. If τ is an n-simplex in $\mathcal{R}$ which does not contain v, then $\tau \in \mathcal{S}$. If τ is an n-simplex in $\mathcal{R}$ which contains v, then we will prove that $\operatorname{ind}(\tau)$ is less than the index of the n-simplex in $\mathcal{S}$ which contains τ. From this it easily follows that performing finitely many such starring subdivisions yields a rational subdivision of Δ_n all of whose n-simplices have index 1, and so this subdivision is integral, as desired.

So let $\mathcal{S}$ be a rational subdivision of Δ_n, and let σ be an n-simplex in $\mathcal{S}$ with $\operatorname{ind}(\sigma) > 1$. Let the vertices of σ be $v_1, \ldots, v_{n+1}$. Since $\operatorname{ind}(\sigma) > 1$, there exists $u \in \mathbf{Z}^{n+1}$ and an integer $m > 1$ such that mu lies in the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v_1}, \ldots, \widetilde{v_{n+1}}$ but u does not. Let $a_1, \ldots, a_{n+1}$ be integers such that $mu = \sum_{i=1}^{n+1} a_i \widetilde{v_i}$. For every integer i with $1 \le i \le n+1$ let b_i be an integer such that $0 \le a_i + mb_i < m$. Then

$$m\left(u + \sum_{i=1}^{n+1} b_i \widetilde{v_i}\right) = \sum_{i=1}^{n+1} (a_i + mb_i)\widetilde{v_i}.$$

Because u is not in the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v_1}, \ldots, \widetilde{v_{n+1}}$, it is impossible that $a_i + mb_i = 0$ for $i = 1, \ldots, n+1$. Reindex if necessary so that $a_i + mb_i \ne 0$ if $i \le k$ and $a_i + mb_i = 0$ if $i > k$ for some integer k with $1 \le k \le n+1$. The vector $w = u + \sum_{i=1}^{n+1} b_i \widetilde{v_i}$ is a positive rational linear combination of $\widetilde{v_1}, \ldots, \widetilde{v_k}$, and so $v = \rho(w)$ is a rational point of Δ_n which lies in the open simplex with vertices $v_1, \ldots, v_k$. Since $w \in \mathbf{Z}^{n+1} \cap \mathbf{R}_+^{n+1}$, w is a positive integer mulitple of $\widetilde{v}$. It follows that $\widetilde{v} = \sum_{j=1}^k c_j \widetilde{v_j}$ for rational numbers $c_1, \ldots, c_k$ with $0 < c_j < 1$.

Now let $\mathcal{R}$ be the simplicial complex obtained from $\mathcal{S}$ by starring at v. Let τ be an n-simplex in $\mathcal{R}$ which contains v. Let σ' be the n-simplex in $\mathcal{S}$ which contains τ. Then $v_1, \ldots, v_k$ are vertices of σ', and so the vertices of σ' have the form $v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_{n+1}$. Hence the vertices of τ have the form $v_1, \ldots, \widehat{v_i}, \ldots v_k, v'_{k+1}, \ldots, v'_{n+1}, v$ for some $i \in \{1, \ldots, k\}$. Thus

$$\begin{aligned}
\operatorname{ind}(\tau) &= |\det(\widetilde{v_1}, \ldots, \widehat{\widetilde{v_i}}, \ldots, \widetilde{v_k}, \widetilde{v_{k+1}}', \ldots, \widetilde{v_{n+1}}', \widetilde{v})| \\
&= |\det(\widetilde{v_1}, \ldots, \widehat{\widetilde{v_i}}, \ldots, \widetilde{v_k}, \widetilde{v_{k+1}}', \ldots, \widetilde{v_{n+1}}', \sum_{j=1}^k c_j \widetilde{v_j})| \\
&= |\sum_{j=1}^k c_j \det(\widetilde{v_1}, \ldots, \widehat{\widetilde{v_i}}, \ldots, \widetilde{v_k}, \widetilde{v_{k+1}}', \ldots, \widetilde{v_{n+1}}', \widetilde{v_j})|.
\end{aligned}$$

In the last expression we have a linear combination of k determinants of which all but 1 are 0 because the corresponding matrices have two equal columns. Hence $\mathrm{ind}(\tau) = c_i \mathrm{ind}(\sigma') < \mathrm{ind}(\sigma')$. This completes the proof of Theorem 6.1. ∎

We denote by $PIP^+(\Delta_n)$ the subset of $PIP(\Delta_n)$ of orientation-preserving piecewise integral projective homeomorphisms of Δ_n. Then $PIP^+(\Delta_n)$ is a group, and is a subgroup of $PIP(\Delta_n)$ of index 2.

We next investigate $PIP^+(\Delta_1)$. Let Δ_1' be the 1-simplex in $\mathbf{R}^2$ consisting of points $(t,1)$ with t in the closed interval $[0,1]$. The linear automorphism of $\mathbf{R}^2$ which maps $(1,0)$ to $(1,1)$ and $(0,1)$ to $(0,1)$ induces a homeomorphism from Δ_1 to Δ_1'. This linear automorphism is given by a matrix in $SL(2,\mathbf{Z})$. Thus we can "conjugate" the above discussion leading to the definition of $PIP^+(\Delta_1)$ to Δ_1': we get a group $PIP^+(\Delta_1')$ which is isomorphic to $PIP^+(\Delta_1)$. In so doing, ρ is replaced by the map ρ' that sends (x,y) to $(\frac{x}{y},1)$ if $y \neq 0$ and to $(0,1)$ if $y=0$. An integral projective map for Δ_1' is the composition of ρ' and a function induced by a matrix in $GL(2,\mathbf{Z})$. An integral subsimplex of Δ_1' is a subsimplex of Δ_1' which is homeomorphic to Δ_1' by a Δ_1'-integral projective map.

Now we identify $[0,1]$ with Δ_1' via the map $t \mapsto (t,1)$. Let a be a nonnegative integer and let b,c,d be positive integers such that $a \le b$ and $c \le d$. Then $\gcd(a,b) = 1 = \gcd(c,d)$, $\frac{a}{b} < \frac{c}{d}$ and $[\frac{a}{b},\frac{c}{d}]$ is an integral subsimplex of $[0,1]$ if and only if $ad - bc = -1$. Suppose a,b,c,d are as above such that $[\frac{a}{b},\frac{c}{d}]$ is an integral subsimplex of $[0,1]$. The *left part* of $[\frac{a}{b},\frac{c}{d}]$ is $[\frac{a}{b},\frac{a+c}{b+d}]$ and the *right part* of $[\frac{a}{b},\frac{c}{d}]$ is $[\frac{a+c}{b+d},\frac{c}{d}]$. The left and right parts of $[\frac{a}{b},\frac{c}{d}]$ are integral subsimplices of $[0,1]$. The *tree of integral subsimplices* of $[0,1]$ is the tree $\mathcal{T}'$ with vertices the integral subsimplices of $[0,1]$ and with edges the pairs (I,J) where I and J are integral subsimplices of $[0,1]$ and I is either the left part of J or the right part of J. An edge (I,J) of $\mathcal{T}'$ is a *left edge* if I is the left part of J and is a *right edge* if I is the right part of J.

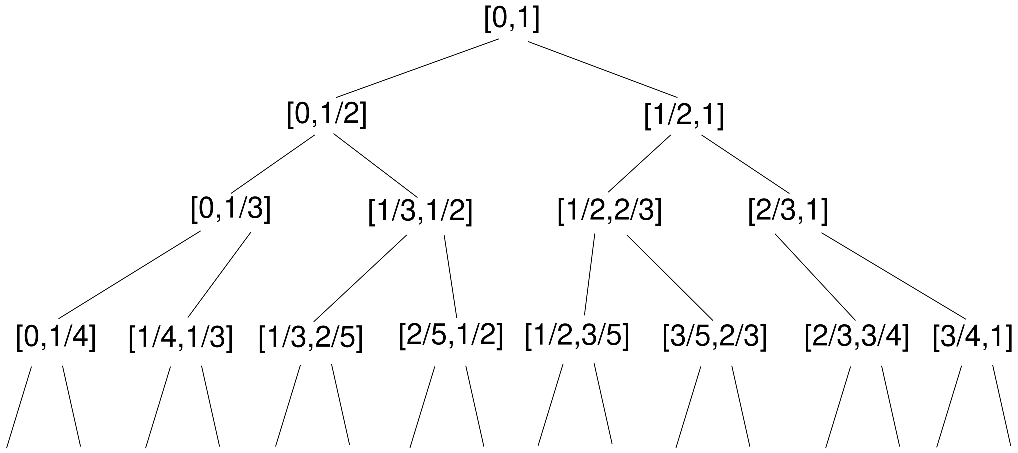

Figure 15. The tree $\mathcal{T}'$ of integral subsimplices of $[0,1]$.

To see that $\mathcal{T}'$ is connected, let a be a nonnegative integer and let b,c,d be positive integers such that $\gcd(a,b) = 1 = \gcd(c,d)$, $[\frac{a}{b},\frac{c}{d}] \neq [0,1]$ and $[\frac{a}{b},\frac{c}{d}]$ is an integral subsimplex of $[0,1]$. First suppose that $a < c$. Let $r = c-a$ and let $s = d-b$. Then $-1 = ad-bc = a(b+s)-b(a+r) = as-br$, so $as = ar+(b-a)r-1$, which implies that $s \ge r$. Furthermore, $[\frac{a}{b},\frac{r}{s}]$ is an integral subsimplex of $[0,1]$ and $[\frac{a}{b},\frac{c}{d}]$ is the left part of $[\frac{a}{b},\frac{r}{s}]$. Now suppose that $a > c$. Let $r = a-c$ and let $s = b-d$. Then $-1 = ad-bc = (c+r)d-(d+s)c = dr-cs$, so $cs = rd+1$ and $s > r$. Furthermore, $[\frac{r}{s},\frac{c}{d}]$ is an integral subsimplex of $[0,1]$ and $[\frac{a}{b},\frac{c}{d}]$ is the right part of $[\frac{r}{s},\frac{c}{d}]$. If $a = c$, then $a = c = 1$, $b = d+1$, and $[\frac{a}{b},\frac{c}{d}]$ is the right part of $[\frac{0}{1},\frac{1}{d}]$. It follows that $\mathcal{T}'$ is connected and hence $\mathcal{T}'$ is an ordered rooted binary tree.

Now we consider integral projective maps for $[0,1]$. It is easy to see that they are given by linear fractional transformations corresponding to matrices in $GL(2,\mathbf{Z})$. Let $[\frac{\alpha}{\beta},\frac{\gamma}{\delta}]$ and $[\frac{a}{b},\frac{c}{d}]$ be integral subsimplices of $[0,1]$ as above. There is a unique integral projective map $f\colon [\frac{\alpha}{\beta},\frac{\gamma}{\delta}] \to [\frac{a}{b},\frac{c}{d}]$ with $f(\frac{\alpha}{\beta}) = \frac{a}{b}$ and $f(\frac{\gamma}{\delta}) = \frac{c}{d}$. The function f is defined by

$$f(t) = \frac{(c\beta - a\delta)t + (a\gamma - c\alpha)}{(d\beta - b\delta)t + (b\gamma - d\alpha)}$$

as a linear fractional transformation and is given by the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}^{-1}$. Since $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

$\begin{pmatrix} a+c \\ b+d \end{pmatrix}$ and $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha+\gamma \\ \beta+\delta \end{pmatrix}$, it follows that $f(\frac{\alpha+\gamma}{\beta+\delta}) = \frac{a+c}{b+d}$, and hence $f\left([\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}]\right) = [\frac{a}{b}, \frac{a+c}{b+d}]$ and $f\left([\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}]\right) = [\frac{a+c}{b+d}, \frac{c}{d}]$. This shows that an integral projective map $f : [\frac{\alpha}{\beta}, \frac{\gamma}{\delta}] \to [\frac{a}{b}, \frac{c}{d}]$ restricts to integral projective maps $f\big| : [\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}] \to [\frac{a}{b}, \frac{a+c}{b+d}]$ and $f\big| : [\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}] \to [\frac{a+c}{b+d}, \frac{c}{d}]$. The converse is also true; if $g_1 : [\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}] \to [\frac{a}{b}, \frac{a+c}{b+d}]$ and $g_2 : [\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}] \to [\frac{a+c}{b+d}, \frac{c}{d}]$ are integral projective maps, then they are the restrictions of an integral projective map $g : [\frac{\alpha}{\beta}, \frac{\gamma}{\delta}] \to [\frac{a}{b}, \frac{c}{d}]$. It follows as in §2 that there is a bijection between $PIP^+(\Delta_1)$ and the set of reduced tree diagrams.

Suppose $f, g \in PIP^+(\Delta_1)$, and let (P, Q) and (R, S) be reduced tree diagrams for f and g. Let Q' be a $\mathcal{T}'$-tree such that $Q \subset Q'$ and $R \subset Q'$. Then there are $\mathcal{T}'$-trees P' and S' such that $P \subset P'$, $S \subset S'$, (P', Q') is a tree diagram for f and (Q', S') is a tree diagram for g. Then (P', S') is a tree diagram for gf. This implies that the group structure for $PIP^+(\Delta_1)$ can be determined by the tree diagrams. Since the tree $\mathcal{T}$ of standard dyadic intervals is isomorphic, as an ordered rooted binary tree, to the tree $\mathcal{T}'$, this proves the following.

Theorem 6.2. $F \cong PIP^+(\Delta_1)$.

We still view S^1 as $[0, 1]$ with the endpoints identified. A *piecewise integral projective (PIP)* homeomorphism of S^1 is a homeomorphism $f : S^1 \to S^1$ such that there is an integral subdivision $\mathcal{S}$ of $[0, 1]$ with $f\big|_\sigma$ integral projective for each simplex σ of $\mathcal{S}$. We denote by $PIP^+(S^1)$ the group of orientation-preserving PIP homeomorphisms of S^1. The proof of Theorem 6.2 also proves Theorem 6.3.

Theorem 6.3. $T \cong PIP^+(S^1)$.

The three functions in $PIP^+(S^1)$ corresponding to A, B and C are the following.

$$A(t) = \begin{cases} \frac{t}{t+1}, & 0 \le t \le \frac{1}{2} \\ \frac{-t+1}{-5t+4}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{2t-1}{t}, & \frac{2}{3} \le t \le 1 \end{cases} \qquad B(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ \frac{3t-1}{4t-1}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{-6t+5}{-11t+9}, & \frac{2}{3} \le t \le \frac{3}{4} \\ \frac{2t-1}{t}, & \frac{3}{4} \le t \le 1 \end{cases} \qquad C(t) = \begin{cases} \frac{-3t+2}{-5t+3}, & 0 \le t \le \frac{1}{2} \\ \frac{2t-1}{t}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{5t-3}{7t-4}, & \frac{2}{3} \le t \le 1 \end{cases}$$

References

[B-S] Bieri, R., Strebel, R., *On groups of PL-homeomorphisms of the real line*, unpublished manuscript.

[B-S] Brin, M. G., Squier, C. C., *Groups of piecewise linear homeomorphisms of the real line*, Invent. math. **79** (1985), 485–498.

[B1] Brown, K. S., *Finiteness properties of groups*, J. Pure App. Algebra **44** (1987), 45–75.

[B2] Brown, K. S., *The geometry of finitely presented infinite simple groups*, preprint.

[B3] Brown, K. S., *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, preprint.

[B-G] Brown, K. S., Geoghegan, R., *An infinite-dimensional torsion-free FP_∞ group*, Invent. math. **77** (1984), 367–381.

[C] Chou, C., *Elementary amenable groups*, Illinois J. Math. **24**, (1980), 396–407.

[D1] Dydak, J., *A simple proof that pointed connected FANR-spaces are regular fundamental retracts of ANR's*, Bull. Polon. Acad. Sci. Ser. Sci. Math. Astronom. Phys. **25** (1977), 55–62.

[D2] Dydak, J., *1-movable continua need not be pointed 1-movable.*, Bull. Polon. Acad. Sci. Ser. Sci. Math. Astronom. Phys. **25** (1977), 485–488.

[F-H] Freyd, P., Heller, A., *Splitting homotopy idempotents, II*, Unpublished manuscript, U. Penn. (1979).

[Ge] Gersten, S. M., *Thompson's group F is not combable*, preprint.

[G-S] Ghys, E., Sergiescu, V., *Sur un groupe remarquable de difféomorphismes du cercle*, Comm. Math. Helv. **62** (1987), 185–239.

[Gr] Greenberg, P., *Projective aspects of the Higman-Thompson Group*, Group Theory from a Geometrical Viewpoint: 21 March - 6 April 1990, ICTP, Trieste, Italy (E. Ghys, A. Haefliger and A. Verjovsky, eds.), World Scientific, Singapore, 1991, pp. 633–644.

[Gr-S] Greenberg, P., Sergiescu, V., *An acyclic extension of the braid group*, Comm. Math. Helv. **66** (1991), 109–138.

[H] Higman, G., *Finitely Presented Infinite Simple Groups*, Notes on Pure Mathematics 8, Australian National University, Canberra (1974).

[P] Paterson, A. L. T., *Amenability*, Mathematical Surveys and Monographs, Number 29, American Mathematical Society, Providence, 1988.

[M-T] McKenzie, R., Thompson, R. J., *An elementary construction of unsolvable word problems in group theory*, Word Problems (W. W. Boone, F. B. Cannonito and R. C. Lyndon, eds.), Studies in Logic and the Foundation of Mathematics, vol. 71, North-Holland, Amsterdam, 1973, pp. 457–478.

[R-S] Rourke, C. P., Sanderson, B. J., *Introduction to Piecewise-Linear Topology*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 69, Springer-Verlag, Berlin Heidelberg New York, 1972.

[S] Stein, M., *Groups of piecewise linear homeomorphisms*, Trans. Amer. Math. Soc. **332** (1992), 477–514.

[T] Thompson, R. J., *Embeddings into finitely generated simple groups which preserve the word problem*, Word Problems II: The Oxford Book (S. I. Adian, W. W. Boone and G. Higman, eds.), Studies in Logic and the Foundations of Mathematics, vol. 95, North-Holland, Amsterdam, 1980, pp. 401–441.

[W] Wagon, S., *The Banach-Tarski Paradox*, Cambridge University Press, Cambridge, 1985.

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