An integrable normal form for water waves in infinite depth

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Abstract: We consider the Birkhoff normal form for the water wave problem posed in a fluid of infinite depth, with the starting point of our analysis a version of the Hamiltonian given by V.E. Zakharov. We verify that in the fourth order normal form, the coefficients vanish for all non-generic resonant terms, and we show that the resulting truncated system is completely integrable. In contrast we show that there are resonant fifth order terms with nonvanishing coefficients, answering to the negative the conjecture of A.I. Dyachenko & Zakharov on the integrability of free surface hydrodynamics. We analyse all solutions of the fourth order integrable system. In particular, the bifurcation structures of standing wave and travelling wave solutions are remarkably explicit due to the normal form.

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1. Introduction

This paper concerns the water wave problem, which is the problem of waves in the free surface of an inviscid and incompressible fluid which satisfies the Euler equations, including the influence of gravity. Motions of the fluid are assumed to be two-dimensional in character and irrotational, and we will take the fluid domain to be 2π periodic in the horizontal variable, and unbounded below. The classical equations for this regime are in terms of the velocity potential φ and the free surface displacement η:

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in} \quad 0 \le x \le 2\pi,\ -\infty < y < \eta(x) \\ \varphi &\to 0 \quad \text{as} \quad y \to -\infty \\ \partial_t \eta &= \partial_y \varphi - \partial_x \eta \partial_x \varphi \\ \partial_t \varphi &= -g\eta - (\nabla\varphi)^2 \end{aligned} \qquad \text{for} \quad y = \eta(x)\ .$$

Over the interval $0 \le x \le 2\pi$ there are two natural cases of boundary conditions that are imposed. The most general boundary conditions are of periodicity of the free surface and velocity potential

$$\eta(x+2\pi, t) = \eta(x,t), \qquad \varphi(x+2\pi, y, t) = \varphi(x,y,t)\ . \tag{1}$$

We will also consider the standing wave boundary conditions

$$\partial_x \eta(0,t) = 0 = \partial_x \eta(\pi, t), \qquad \partial_x \varphi(0,y,t) = 0 = \partial_x \varphi(\pi, y, t)\ , \tag{2}$$

which are appropriate for solutions in a confining basin with fixed vertical sidewalls, as well as for spatially periodic standing wave patterns in a horizontally infinite fluid body.

This problem can be formulated as a Hamiltonian system, with a Hamiltonian function given originally by Zakharov [12]. We will exploit this fact in a convenient form given in Craig [3] and Craig & Sulem [6]. The equations of motion are restated

$$\partial_t \eta = \partial_\xi H \qquad \partial_t \xi = -\partial_\eta H\ , \tag{3}$$

with canonical conjugate variables $(\eta(x), \xi(x))$, where $\xi(x) = \varphi(x, \eta(x))$, and Hamiltonian function the total energy

$$\begin{aligned} H(\eta,\xi) &= \tfrac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\eta(x)} (\nabla\varphi)^2\, dy dx + \tfrac{1}{2}\int_0^{2\pi} g\eta^2\, dx \\ &= \tfrac{1}{2}\int_0^{2\pi} \xi G(\eta)\xi + g\eta^2\, dx\ . \end{aligned} \tag{4}$$

The kinetic energy, or Dirichlet integral in (4) is expressed in terms of the Dirichlet-Neumann operator $G(\eta)\xi \; dx = \nabla\varphi \cdot \vec{N} \; dS$, where φ is the harmonic extension into the fluid domain of the boundary values given by ξ.

It is known that $H(\eta,\xi) = H_2(\eta,\xi) + H_3(\eta,\xi) + H_4(\eta,\xi) + \dots$, has a Taylor series expansion about the equilibrium $u = (\eta,\xi) = 0$, and that $\partial_u^2 H(0) \geq 0$ so the equilibrium has the character of an elliptic stationary point. It is therefore natural to understand the system more completely by transformations to Birkhoff normal form to some given order. In Craig [4] it is shown that the transformation to third order Birkhoff normal form is well defined in a neighborhood of $(\eta,\xi) = 0$, in Banach spaces of functions analytic in a strip. In this article we will focus instead on the formal aspects of the transformation theory, and will leave questions of convergence aside.

The main results of this paper concern the character of approximations to the water wave system (3) through truncations of the Birkhoff normal form. In his recent seminar talk at the MSRI in Berkeley [13], V.E. Zakharov announced that the coefficients of all fourth order non-generic resonant terms vanish. (this work is reported in Dyachenko & Zakharov [7]). We verify below using the formulation (3, 4). It does not follow from this that the resulting truncated system of equations is integrable, since for each wavenumber k there are two spatial modes in $1:1$ resonance. In particular the usual action integrals of the linearized equation are not constants of motion of the nonlinear system. We show below that despite these resonances the equations are completely integrable, and we give a complete set of action-angle variables, taking account of degeneracies. From this analysis it follows that all solutions to the initial value problem for this fourth order approximation to the water wave Hamiltonian are almost-periodic in time. The evolution of resonant pairs of spatial modes is governed by a two degree of freedom system, whose solutions correspond to either time periodic standing waves, time periodic travelling waves, or for the most part, time quasi-periodic motions comprising two frequencies. The general solution consists of a nonlinear superposition of an arbitrary and possibly infinite number of these solutions, and is in general almost-periodic in time, with easily determined almost-frequencies. In this paper we exhibit a number of solutions of this integrable approximation to nonlinear deep water waves, and we want to emphasize that these are not numerical results but are obtained through exact quadrature.

Given the unexpected cancellations in the resonant coefficients which result in an integrable fourth order Birkhoff normal form, it is natural to conjecture (as do Dyachenko and Zakharov) that resonant term cancellations occur for all order normal forms, which would be strong evidence for the integrability of the full deep water wave problem. We are able to continue our calculations on this formal level to the fifth order Birkhoff normal form, where we show that the expectation of integrability in the above sense fails. This failure occurs because there are fifth order resonances (k,ℓ,m,n,o) such that for some choice of signs,

$$k+\ell+m+n+o = 0 \; , \quad \sqrt{|k|} \pm \sqrt{|\ell|} \pm \sqrt{|m|} \pm \sqrt{|n|} \pm \sqrt{|o|} = 0 \; ,$$

with nonzero coefficients of the associated resonant term in the water wave Hamiltonian. We exhibit certain fifth order resonant terms whose coefficients after the fifth order normal forms transformation do not vanish. To pursue this breakdown of integrability, we classify (to some extent) the possible resonant fifth order quintuples, which is in fact a question in number theory.

The unexpectedly simple form of the fourth order Birkhoff normal form approximation to the water waves system is easily amenable to the analysis of standing waves, travelling waves and their bifurcations. In the case of standing waves we recover the bifurcation analysis of simple branches for nonresonant modes, described earlier in Penny & Price [10] and Tadjbakhsh & Keller [11]. We carry the analysis further to the second and higher bifurcations. As these equations describe the weakly nonlinear regime, the most interesting two-mode bifurcations involve wave numbers k and ℓ such that $k/\ell = p^2$, where p is an integer. In this case there is bifurcation from zero of families of time periodic solutions, parameterized by tori, with frequency near $\omega(k) = p\omega(\ell)$. All standing waves are neutrally stable under perturbations in the class which preserve the Neumann boundary conditions (2), but are unstable under general perturbations satisfying spatially periodic boundary conditions (1).

The Birkhoff normal form reduced the study of the travelling wave bifurcation structure to a family of linear problems in the positive orthant of the space of action variables, which in principle has an exact solution. We work out the complete details of bifurcation points and connectivity of bifurcation branches for travelling waves which involve principally one and two spatial modes (respectively, class 1 and class 2 solutions). In our analysis these spatial modes can be for any two wave numbers. Two mode branches are again parameterized by tori, and most solutions on a two mode branch are not spatially symmetric through any vertical axis (for a discussion of symmetry of spatially periodic travelling water waves see Garabedian [8] and Zufiria [14]). Our analysis is less complete in this respect than the latter reference, as we are working with a finite order of normal form, which limits the validity of approximation to the weakly nonlinear regime.

There are other nonlinear problems that are amenable to analysis via infinite dimensional normal forms transformations. Recent examples include Kuksin & Pöschel [9], who use a globally convergent fourth order normal form for the nonlinear Klein-Gordon equation in order to satisfy the nondegeneracy hypotheses of S. Kuksin's infinite dimensional version of KAM theory. A second is in the work of Craig, Kuksin & Wayne [5], where a partial normal form is used in a construction of hyperbolic periodic orbits (whiskered circles) for the nonlinear wave equation and nonlinear Schrödinger equation. We expect that the techniques developed along these lines will be useful for other problems, including further studies of water waves in finite depth, and in considerations of other Hamiltonian systems with infinitely many degrees of freedom.

2. The normal form

In this section we demonstrate how to compute the degree four normal form and show that it is completely integrable. The starting point is the Hamiltonian dynamical system given in (3,4). The Dirichlet-Neumann operator, $G(\eta)$, acts on ξ linearly and may be written in a Taylor series about $\eta = 0$, $G(\eta) = G_0 + \sum_{k=1}^{\infty} G_k(\eta)$. The terms $G_i(\eta)$ can be computed recursively (see Craig and Sulem [6]); we will be using the first few, which are

$$\begin{aligned}
G_0 &= D\tanh(hD) \\
G_1(\eta) &= D(\eta - \tanh(hD)\eta\tanh(hD))D \\
G_2(\eta) &= -\tfrac{1}{2}D(D\eta^2\tanh(hD) + \tanh(hD)\eta^2 D \\
&\quad -2\tanh(hD)\eta D\tanh(hD)\eta\tanh(hD))D \\
G_3(\eta) &= \tfrac{1}{6}D\eta^3 D^3 - \tfrac{1}{6}G_0\eta^3 D^3\tanh(hD) \\
&\quad -\tfrac{1}{2}G_1(\eta)\eta^2 D^2 - G_2(\eta)\eta D\tanh(hD)\ ,
\end{aligned}$$

where h is the depth of the fluid and $D \equiv -i\partial_x$.

To compute with these expressions, we write η and ξ as Fourier series

$$\eta(x) = \frac{1}{\sqrt{2\pi}}\sum_k \eta_k e^{ikx} \qquad \xi(x) = \frac{1}{\sqrt{2\pi}}\sum_k \xi_k e^{ikx}\ .$$

If we now consider the deep water limit, which is the limit as $h \to \infty$, the Hamiltonian H written in terms of $\hat{\eta} = (\ldots, \eta_{-1}, \eta_0, \eta_1, \ldots)$ and $\hat{\xi} = (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots)$ is given by

$$\begin{aligned}
H(\hat{\eta}, \hat{\xi}) &= \frac{1}{2}\sum_{k+l=0} g\eta_k\eta_l + \frac{1}{2}\sum_{k+l=0}|k|\xi_k\xi_l \\
&\quad + \frac{1}{2\sqrt{2\pi}}\sum_{k+l+m=0}(-kl - |kl|)\xi_k\xi_l\eta_m \\
&\quad - \frac{1}{8\pi}\sum_{k+l+m+n=0}|kl|(|l| + |k| - 2|k+m|)\xi_k\xi_l\eta_m\eta_n \\
&\quad + R_5\ ,
\end{aligned}$$

where R_5 is the Taylor remainder term in η, ξ. The transformation to complex coordinates is given by

$$z_k = \frac{\eta_k + \eta_{-k}}{\sqrt{2\omega_k/g}} + j\frac{\xi_k + \xi_{-k}}{\sqrt{2g/\omega_k}} \qquad w_k = \frac{\eta_k - \eta_{-k}}{i\sqrt{2\omega_k/g}} + j\frac{\xi_k - \xi_{-k}}{i\sqrt{2g/\omega_k}}\ ,$$

for $k \neq 0$, where $\omega_k = \sqrt{g|k|}$ and j is a new complex involute associated with z and w. Reality conditions on the solutions η and ξ force z and w to be independent of i so the introduction of the letter j to represent the symplectic structure is primarily for convenience. We also note that by definition $z_{-k} = z_k$ and $w_{-k} = -w_k$. In these new coordinates, the Hamiltonian is written

$$
\begin{aligned}
H(z,w) &= \frac{1}{2}\sum_{k>0}\sqrt{gk}(z_k\bar{z}_k + w_k\bar{w}_k) \\
&+\frac{\sqrt[4]{g}}{32\sqrt{\pi}}\sum_{k,l>0}\sqrt[4]{k^3l^3(k+l)}\,(\alpha_k\alpha_l\beta^*_{k+l} + \alpha^*_k\alpha^*_l\beta_{k+l}) \\
&+\frac{1}{512\pi}\sum_{k,l,m}\sqrt[4]{|k^3l^3(m-k)(m-l)|}\,(|l|+|k|-2|k+m|)\alpha_k\alpha^*_l\beta_{m-k}\beta^*_{m-l} \\
&+R_5 \ ,
\end{aligned}
\tag{5}
$$

where * and $^-$ denote complex conjugation with respect to i and j, respectively, and

$$\alpha_k = z_k - \bar{z}_k + i(w_k - \bar{w}_k) \qquad \beta_k = z_k + \bar{z}_k + i(w_k + \bar{w}_k) \ .$$

In complex coordinates the Poisson bracket is defined by

$$[P,Q] = -2j\sum_{k>0}(\frac{\partial P}{\partial \bar{z}_k}\frac{\partial Q}{\partial z_k} - \frac{\partial P}{\partial z_k}\frac{\partial Q}{\partial \bar{z}_k} + \frac{\partial P}{\partial \bar{w}_k}\frac{\partial Q}{\partial w_k} - \frac{\partial P}{\partial w_k}\frac{\partial Q}{\partial \bar{w}_k}) \ .$$

Two elementary quantities conserved by the evolution of (3) are the mass η_0 and ξ_0, which is a result of the calculation

$$[H,\eta_0] = 0, \qquad [H,\xi_0] = 0 \ .$$

Thus, without any loss of generality we may restrict our attention to the space $\{(\eta_0,\xi_0) = 0\}$, which is preserved under the evolution. The equations of motion (3) may be written as

$$\partial_t z_k \ = [H(z,w), z_k] \qquad \partial_t w_k \ = [H(z,w), w_k] \ , \tag{6}$$

for $k \geq 1$.

Normal forms transformations are used to eliminate unnecessary terms in the Hamiltonian $H(z,w)$. By the standard theory (see, for example, Churchill, Kummer, and Rod [2]) on a formal level all nonresonant terms may be eliminated. We quickly summarize the result.

Let $\mathcal{P}_j$ denote the infinite-dimensional vector space of all homogeneous polynomials of degree j in the variables $\{z_k, \bar{z}_k, w_k, \bar{w}_k\}_k$. We write $\mathcal{P} = \bigoplus \mathcal{P}_j$ for the space of all polynomials. A monomial will be written as $z^a\bar{z}^{\bar{a}}w^b\bar{w}^{\bar{b}}$ where $a, \bar{a}, b, \bar{b}$ are multi-indices, (so that a_k is the power of z_k in the monomial and similarly for $\bar{a}_k, b_k$, and $\bar{b}_k$). We write $|a| \equiv \sum_{k>0} a_k$ so the degree of the monomial $z^a\bar{z}^{\bar{a}}w^b\bar{w}^{\bar{b}}$ is given by $|a| + |\bar{a}| + |b| + |\bar{b}|$.

The operator ad_{H_2} is defined by $\mathrm{ad}_{H_2}(\cdot) = [H_2, \cdot]$, which acts diagonally on $\mathcal{P}_j$ in the basis of monomials:

$$\mathrm{ad}_{H_2}(z^a\bar{z}^{\bar{a}}w^b\bar{w}^{\bar{b}}) = -j\sqrt{g}(\sum_{k>0}\sqrt{k}(a_k - \bar{a}_k + b_k - \bar{b}_k))z^a\bar{z}^{\bar{a}}w^b\bar{w}^{\bar{b}} \ .$$

A monomial $N \in \mathcal{P}$ is called resonant if $[H_2, N] = 0$; that is, for $N = z^a \bar{z}^{\bar{a}} w^b \bar{w}^{\bar{b}}$, N is resonant if and only if $\sum_{k>0} \sqrt{k}(a_k - \bar{a}_k + b_k - \bar{b}_k) = 0$. The vector spaces $\mathcal{P}$ and $\mathcal{P}_j$ have splittings into vector subspaces; one contains polynomials whose terms are all resonant and the other polynomials whose terms are all nonresonant. We denote these spaces by $\mathcal{F}$ and $\mathcal{F}_j$ for the nonresonant polynomials and $\mathcal{G}$ and $\mathcal{G}_j$ for the resonant polynomials. Let $\Pi_{\mathcal{G}}$ and $\Pi_{\mathcal{G}_j}$ denote the projection operators onto the subspaces $\mathcal{G}$ and $\mathcal{G}_j$, respectively. We use similar notation for projection onto the $\mathcal{F}$ and $\mathcal{F}_j$ subspaces.

The theory of normal forms produces a series of near-identity transformations which remove the nonresonant terms in the Hamiltonian degree by degree. We write the Hamiltonian of (5) as

$$H(z, w) = H_2 + H_3 + H_4 + H_5 + R_6 \ ,$$

where H_2, H_3, H_4, H_5 are the quadratic, cubic, quartic, and quintic terms, respectively, and R_6 represents the degree 6 and higher terms. We now perform the first two normal form transformations, eliminating nonresonant terms of degree 3 and 4. Let $F_3 = \Pi_{\mathcal{F}} H_3$ and $G_3 = \Pi_{\mathcal{G}} H_3$. Let $K_3 = \mathrm{ad}_{H_2}^{-1}(F_3)$ and then the time one map of the Hamiltonian flow generated by K_3 is a symplectic transformation which transforms the Hamiltonian to $\exp(\mathrm{ad}_{K_3})(H)$. The process can be repeated at each degree.

Lemma 1 *There are no resonant cubic terms in $H(z, w)$, i.e., $G_3 = \Pi_{\mathcal{G}} H_3 = 0$.*

Proof A resonant cubic monomial must have positive integer wave numbers which satisfy the following equations

$$\begin{gathered} k \pm l \pm m = 0 \\ \sqrt{k} \pm \sqrt{l} \pm \sqrt{m} = 0 \ . \end{gathered} \tag{7}$$

The first equation is required for a term to be in the cubic part of the Hamiltonian while the second one is the resonance condition. It is an elementary and well known fact that (7) has no positive integer solutions. □

The first normal form transformation thus gives

$$\begin{aligned} \hat{H} &= H_2 + (H_4 + \tfrac{1}{2}[K_3, H_3]) + (H_5 + [K_3, H_4] + \tfrac{1}{3}[K_3, [K_3, H_3]]) + \hat{R}_6 \\ &\equiv H_2 + \hat{H}_4 + \hat{H}_5 + \hat{R}_6 \ . \end{aligned}$$

We now turn our attention to removing quartic terms in the Hamiltonian. The degree four resonant condition is given by

$$\begin{gathered} k \pm l \pm m \pm n = 0 \\ \sqrt{k} \pm \sqrt{l} \pm \sqrt{m} \pm \sqrt{n} = 0 \ , \end{gathered} \tag{8}$$

for some choice of $\pm$ signs.

Lemma 2 *Every positive integer solution to (8) is either of the form*

$$(p,p,q,q)$$

or of the form

$$\left(q(p^2+p+1)^2, qp^2(p+1)^2, q(p+1)^2, qp^2\right) \ ,$$

where p and q are positive integers.

The proof of this is also straightforward. We call resonances of the first kind *generic resonances* and those of the second kind *Benjamin-Feir resonances.* The first result, announced previously by Zakharov [13] and Dyachenko & Zakharov [7], regarding the quartic normal form is stated in the following theorem.

Theorem 3 *The normal form transformation removing the cubic terms of H, (5), also removes the Benjamin-Feir resonant quartic terms.*

Proof From scaling considerations, it is sufficient to examine the resonant quartic terms with modes $p^2, (p+1)^2, p^2(p+1)^2, (p^2+p+1)^2$ in $\hat{H}_4 = H_4 + \frac{1}{2}[K_3, H_3]$. Although, H_4, H_3 and $K_3 = \mathrm{ad}_{H_2}^{-1}(H_3)$ are infinite sums, only a finite number of the terms contribute to any particular resonance. We must identify this finite set.

The Benjamin-Feir resonance terms for fixed p which occur in H_4 are contained in

$$\Pi_{\mathcal{G}}\left(\frac{1}{512\pi}\sum_{(k,l,m)\in\mathcal{S}}\sqrt[4]{|k^3l^3(m-k)(m-l)|}\,(|l|+|k|-2|k+m|)\alpha_k\alpha_l^*\beta_{m-k}\beta_{m-l}^*\right) \ ,$$

where the finite set of indices is

$$\mathcal{S} = \left\{(k,l,m) | \{|k|,|l|,|m-l|,|m-k|\} = \{p^2,(p+1)^2,p^2(p+1)^2,(p^2+p+1)^2\}\right\} \ .$$

It is substantially more complicated to determine the terms in H_3 which contribute through $\frac{1}{2}[\mathrm{ad}_{H_2}^{-1}(H_3), H_3]$. We consider cubic monomials which can combine through the Poisson bracket to give the desired four wave numbers. Thus two wave numbers must come from each cubic and the third index of each cubic must be the same wave number. A careful inspection shows that all possibilities are included if we consider the following finite sum:

$$\frac{\sqrt[4]{g}}{32\sqrt{\pi}}\sum_{(k,l)\in\mathcal{T}}\sqrt[4]{k^3l^3(k+l)}\,(\alpha_k\alpha_l\beta_{k+l}^* + \alpha_k^*\alpha_l^*\beta_{k+l}) \ ,$$

where

$$\begin{aligned}\mathcal{T} = \ &\{(p^2,(p+1)^2),((p+1)^2,p^2),(p^2+(p+1)^2,p^2(p+1)^2),(p^2(p+1)^2,p^2+(p+1)^2),\\ &(p^2,p^2(p+1)^2),(p^2(p+1)^2,p^2),((p+1)^2,p^2+p^2(p+1)^2),\\ &(p^2+p^2(p+1)^2,(p+1)^2),(p^2,(p+1)^2+p^2(p+1)^2),((p+1)^2+p^2(p+1)^2,p^2),\\ &((p+1)^2,p^2(p+1)^2),(p^2(p+1)^2,(p+1)^2)\} \ .\end{aligned}$$

We use Maple [1] to actually perform the rather lengthy calculations which do indeed show that there are no Benjamin-Feir resonant terms in the sum $H_4 + \frac{1}{2}[K_3, H_3]$. $\square$

This implies that all degree four terms except the generic resonant ones may be removed from the Hamiltonian by normal form transformations. Let F_4 be the nonresonant terms of $\hat{H}_4$ and let $K_4 = \mathrm{ad}_{H_2}^{-1}(F_4)$ be the flow to generate the next transformation. This gives the result

$$\begin{aligned}\hat{\hat{H}} &= H_2 + (\hat{H}_4 - F_4) + \hat{H}_5 + \hat{\hat{R}}_6 \\ &\equiv H_2 + \hat{\hat{H}}_4 + \hat{H}_5 + \hat{\hat{R}}_6 \ .\end{aligned}$$

Performing these straightforward, but rather lengthy, calculations allows us to write the Hamiltonian

$$\hat{\hat{H}}(z,w) = \sum_{k>0}\left(\sqrt{gk}I_1(k) - \frac{k^3}{8\pi}(I_1^2(k) - 3I_2^2(k))\right) + \frac{1}{\pi}\sum_{0<k<l} k^2 l I_2(k) I_2(l) + \hat{H}_5 + \hat{\hat{R}}_6 \ ,$$

with

$$I_1(k) = (z_k\bar{z}_k + w_k\bar{w}_k)/2 \qquad I_2(k) = (z_k\bar{w}_k - \bar{z}_k w_k)/2j \ . \tag{9}$$

This expression is the Hamiltonian after two normal form transformations. The lowest degree nonresonant terms are contained in $\hat{H}_5$, which are of fifth degree.

Momentum is conserved in the water wave problem, where the momentum integral is

$$M(\eta,\xi) = \int_0^{2\pi} \xi\partial_x\eta dx \ .$$

Transforming using Fourier series and then to the z and w variables, we find that

$$M(z,w) = \sum_{k>0} kI_2(k) \ . \tag{10}$$

We will now transform this quantity by the two normal form transformations that have been applied to the Hamiltonian. This will be used in the following section when we examine travelling waves.

Proposition 4 $\hat{\hat{M}} = M$.

Proof The normal form transformations give first $\hat{M} = \exp(\mathrm{ad}_{K_3})(M)$ and then $\hat{\hat{M}} = \exp(\mathrm{ad}_{K_4})(\hat{M})$. The following three identities, which are not hard to verify, are useful for applying the normal form transformations to M:

1. $[M, H_j] = 0$.
2. $[I_2(k), \mathrm{ad}_{H_2}^{-1}(F)] = \mathrm{ad}_{H_2}^{-1}([I_2(k), F])$ for all k and $F \in \mathcal{F}$.
3. $[M, \Pi_{\mathcal{G}_j} P] = \Pi_{\mathcal{G}_j}[M, P]$ for all $P \in \mathcal{P}$, and consequently $[M, G_j] = [M, F_j] = 0$.

We calculate

$$\begin{aligned}[K_3, M] &= [\mathrm{ad}_{H_2}^{-1}(F_3), M] \\ &= \mathrm{ad}_{H_2}^{-1}([F_3, M]) \\ &= 0\ .\end{aligned}$$

That is, $\exp(\mathrm{ad}_{K_3})$ acts on M like the identity, leaving $\hat{M} = M$. We now calculate

$$\begin{aligned}[K_4, \hat{M}] &= [\mathrm{ad}_{H_2}^{-1}(\hat{F}_4), M] \\ &= \mathrm{ad}_{H_2}^{-1}([\hat{F}_4, M]) \\ &= \mathrm{ad}_{H_2}^{-1}([\Pi_{\mathcal{F}_4}\hat{H}_4, M]) \\ &= \mathrm{ad}_{H_2}^{-1}(\Pi_{\mathcal{F}_4}([\hat{H}_4, M])) \\ &= \mathrm{ad}_{H_2}^{-1}(\Pi_{\mathcal{F}_4}([H_4 + [K_3, H_3]/2, M])) \\ &= \mathrm{ad}_{H_2}^{-1}(\Pi_{\mathcal{F}_4}([H_4, M] + [[K_3, H_3]/2, M])) \\ &= \mathrm{ad}_{H_2}^{-1}(\Pi_{\mathcal{F}_4}([[K_3, H_3], M]))/2 \\ &= \mathrm{ad}_{H_2}^{-1}(\Pi_{\mathcal{F}_4}(-[[H_3, M], K_3] - [[M, K_3], H_3]))/2 \\ &= 0\ .\end{aligned}$$

So $\exp(\mathrm{ad}_{K_4})$ also acts on $\hat{M}$ like the identity, leaving $\hat{\hat{M}} = \hat{M} = M$. □

The quartic normal form which gives the Hamiltonian system that is the subject of the remainder of this paper is the truncation of $\hat{\hat{H}}$ to degree four, which we denote as

$$\mathcal{H} = \sum_{k>0} \left(\sqrt{gk} I_1(k) - \frac{k^3}{8\pi}(I_1^2(k) - 3I_2^2(k)) \right) + \frac{1}{\pi} \sum_{0<k<l} k^2 l I_2(k) I_2(l)\ . \tag{11}$$

The flow is generated via the Poisson bracket,

$$\begin{aligned}\partial_t z_k &= [\mathcal{H}, z_k] = -2j\partial\mathcal{H}/\partial\bar{z}_k \\ \partial_t w_k &= [\mathcal{H}, w_k] = -2j\partial\mathcal{H}/\partial\bar{w}_k\ .\end{aligned} \tag{12}$$

To recover the water wave surface from a solution $(z_k(t), w_k(t))_{k=1}^{\infty}$ of (12), one inverts the two normal form transformations above, to recover the Fourier coefficients of the surface $\eta(x,t)$ and the boundary values of the velocity potential $\xi(x,t)$. In fact, the inversion need only be accurate up to $O(|z|^4, |w|^4)$ in order that the surface dynamics be consistently represented by the solution of (12).

Theorem 5 *The Hamiltonian dynamical system defined by (11) and (12) is completely integrable.*

Proof With $I_1(k)$ and $I_2(k)$ as given in (9), we find that $[I_1(k), I_1(l)] = 0$, $[I_1(k), I_2(l)] = 0$, and $[I_2(k), I_2(l)] = 0$. From properties of the Poisson bracket and the fact that $\mathcal{H}$ is dependent only on z and w through I_1 and I_2, we see that $[\mathcal{H}, I_1(k)] = [\mathcal{H}, I_2(k)] = 0$. Thus $\{I_1(k), I_2(k)\}_{k>0}$ are a set of Poisson commuting integrals for $\mathcal{H}$. They are also independent. □

We can determine classical action-angle coordinates. The actions are given by the set $\{I_1(k), I_2(k)\}_{k>0}$ and the corresponding angles are defined by

$$\begin{aligned}\varphi_1(k) &= -\frac{1}{2}\tan^{-1}\left(\frac{\mathrm{Im}(z_k^2 + w_k^2)}{\mathrm{Re}(z_k^2 + w_k^2)}\right)\\ \varphi_2(k) &= -\frac{1}{2}\tan^{-1}\left(\frac{z_k\bar{w}_k + \bar{z}_k w_k}{z_k\bar{z}_k - w_k\bar{w}_k}\right) .\end{aligned}$$

The time evolution of the angles is given by

$$\begin{aligned}\partial_t\varphi_1(k) &= \partial\mathcal{H}/\partial I_1(k)\\ &= \sqrt{gk} - \frac{k^3}{4\pi}I_1(k) \equiv \Omega_1(k)\\ \partial_t\varphi_2(k) &= \partial\mathcal{H}/\partial I_2(k)\\ &= \frac{3k^3}{4\pi}I_2(k) + \frac{k}{\pi}\sum_{0<l<k} l^2 I_2(l) + \frac{k^2}{\pi}\sum_{k<l} l I_2(l) \equiv \Omega_2(k) .\end{aligned}$$

The time evolution of any set of initial conditions is thus explicit. We give the expressions for z_k and w_k in terms of the action-angle variables:

$$\begin{aligned}z_k &= \sqrt{\frac{I_1(k) - I_2(k)}{2}}e^{-j(\varphi_1(k)-\varphi_2(k))} + \sqrt{\frac{I_1(k) + I_2(k)}{2}}e^{-j(\varphi_1(k)+\varphi_2(k))}\\ w_k &= \sqrt{\frac{I_1(k) - I_2(k)}{2}}e^{-j(\varphi_1(k)-\varphi_2(k)-\pi/2)} - \sqrt{\frac{I_1(k) + I_2(k)}{2}}e^{-j(\varphi_1(k)+\varphi_2(k)-\pi/2)} .\end{aligned}$$

3. One degree further

A natural question to consider is whether the normal form remains integrable when carried out to higher degrees, despite the presence of nontrivial resonances. In this section we show that integrability with respect to the action variables $\{I_1(k), I_2(k)\}_{k>0}$ fails already at degree five, due to the occurrence of nonzero coefficients of resonant terms in the degree five normal form. The conditions for a degree five resonance are that for integers (k, l, m, n, o),

$$\begin{gathered}k \pm l \pm m \pm n \pm o = 0\\ \sqrt{k} \pm \sqrt{l} \pm \sqrt{m} \pm \sqrt{n} \pm \sqrt{o} = 0 ,\end{gathered}$$

again for some choice of $\pm$ signs. We note first that all the degree five resonances are nongeneric. A resonance may be classified by the number of different wave numbers it contains. It is an elementary fact that there are no one-mode degree five resonances. Every two-modes degree five resonance has the form

$$(p, p, p, p, 4p) \ ,$$

which is the simplest for the analysis of coefficients in the Birkhoff normal form. Using similar techniques as we used to examine the degree four normal form, we can compute the coefficients of these resonant terms. After the first two normal form transformations have been applied, the quintic part of the Hamiltonian is given by $\hat{H}_5 = H_5 + [K_3, H_4] + \frac{1}{3}[K_3, [K_3, H_3]]$.

Theorem 6 *The terms $[K_3, H_4] + \frac{1}{3}[K_3, [K_3, H_3]]$ due to the normal form transformations do not contribute to the $(p, p, p, p, 4p)$ resonant terms in $\hat{H}_5$. Thus the coefficients of the resonant terms come solely from H_5, and are calculated to be*

$$\begin{aligned} \tfrac{p^4}{12\pi}\sqrt[4]{\tfrac{4p}{g\pi^2}}\ Re(\quad & z_{4p}(z_p\bar{z}_p^3 + w_p\bar{w}_p^3 - 3\bar{z}_p\bar{w}_p(z_p\bar{w}_p + \bar{z}_p w_p)) \\ & -w_{4p}(z_p\bar{w}_p^3 - \bar{z}_p^3 w_p - 3\bar{z}_p\bar{w}_p(z_p\bar{z}_p - w_p\bar{w}_p)) \ ) \ . \end{aligned}$$

The conclusion is that the degree five truncation of the water wave Hamiltonian in normal form is not integrable, at least not with respect to the natural coordinates $\{I_1(k), I_2(k)\}_{k>0}$ adapted to the quadratic part of the Hamiltonian.

The integrability of the deep water problem at fourth but not at fifth degree has some unexpected consequences in perturbation analysis of small amplitude solutions. For example, periodic solutions of the degree four Birkhoff normal form occur in families parametrized by tori (the angles of the integrable system), which are expected to break up under the addition of the nonintegrable higher order corrections. These perturbations start at fifth degree, and therefore one can expect among other things that the periodic solutions to occur in multiples of 5, and their frequencies to approximate $1/5$ the perturbed frequency. We will not discuss this issue in the present paper, however.

We will finish this section with a tabulation of the other possible degree five resonances for the deep water wave problem, which is the analog of Lemma 2. Needless to say, it is somewhat more complicated than the two parameter family of the degree four resonances.

Lemma 7 *All three-modes degree five resonances have one of the following forms:*

$$\begin{gathered} (p, p, 2p, 2p, 4p) \\ (p, p, 3p, 3p, 4p) \\ (p, p, 9p, 9p, 16p) \\ (9p, 9p, 9p, 16p, 25p) \ . \end{gathered}$$

There are a large number of families of four-modes degree five resonances. To aid in the enumeration of these, we define some triples of positive integers, which may be thought of as generalized Pythagorean triples. Let

$$
\begin{aligned}
P_0 &= \{(x,y,z)|\ x<y<z, x^2+y^2=z^2\} \\
\tilde{P}_0 &= \{(x,y,z)\in P_0|\ x<3y/4\} \\
\tilde{\tilde{P}}_0 &= \{(x,y,z)\in P_0|\ x>3y/4\} \\
P_1 &= \{(x,y,z)|\ x<y<z, x^2+xz=y^2\} \\
P_2 &= \{(x,y,z)|\ x<y<z, z^2-xz=y^2\} \\
P_3 &= \{(x,y,z)|\ x<y<z, xy+y^2=z^2\} \\
P_4 &= \{(x,y,z)|\ x<y<z, z^2-yz=x^2\} \\
\tilde{P}_4 &= \{(x,y,z)\in P_4|\ y>3z/4\} \\
\tilde{\tilde{P}}_4 &= \{(x,y,z)\in P_4|\ y<3z/4\} \\
P_5 &= \{(x,y,z)|\ x<y<z, x^2+xy+y^2=z^2\} \\
\tilde{P}_5 &= \{(x,y,z)\in P_5|\ (x-y+z) \bmod 3=0\} \\
\tilde{\tilde{P}}_5 &= \{(x,y,z)\in P_5|\ (-x+y+z) \bmod 3=0, y<(1+\sqrt{2})x/2\} \\
P_6 &= \{(x,y,z)|\ x<y<z, x^2-xz+z^2=y^2\}\ . \\
\tilde{P}_6 &= \{(x,y,z)\in P_6|\ (x-y+z) \bmod 3=0\}\ .
\end{aligned}
$$

Lemma 8 *All four-modes degree five resonances have one of the following forms:*

$$
\begin{array}{cl}
(x^2,y^2,z^2,(xy+z(x+y+z))/2,(xy+z(x+y+z))/2) & \textit{with } (x,y,z)\in P_0 \\
(x^2,y^2,xy+y^2,xy+y^2,x^2+2xy+y^2) & \textit{with } x<y \\
(x^2,z^2,(x+y+z)^2,(x+y+z)^2,2(x+y+z)^2+x^2-z^2) & \textit{with } (x,y,z)\in P_1 \\
(x^2,y^2,x^2+xy,x^2+xy,x^2+2xy+y^2) & \textit{with } x<y<\frac{1+\sqrt{5}}{2}x \\
(x^2,z^2,(z+y-x)^2,(z+y-x)^2,2(z+y-x)^2+z^2-x^2) & \textit{with } (x,y,z)\in P_2 \\
(x^2,y^2,(x+y+z)^2,(x+y+z)^2,2(x+y+z)^2+y^2-x^2) & \textit{with } (x,y,z)\in P_3 \\
(x^2,y^2,x^2+xy+y^2,x^2+xy+y^2,x^2+2xy+y^2) & \textit{with } x<y \\
(x^4,y^4,(x^2+xy+y^2)^2,(x^2+xy+y^2)^2,(x+y)^4) & \textit{with } x<y \\
(x^2,xy,xy,y^2,x^2+2xy+y^2) & \textit{with } x<y \\
(y^2,(x-y+z)^2,(x-y+z)^2,z^2,z^2+2(x-y+z)^2-y^2) & \textit{with } (x,y,z)\in \tilde{\tilde{P}}_4 \\
(x^2,x^2+xy,x^2+xy,y^2,x^2+2xy+y^2) & \textit{with } \frac{1+\sqrt{5}}{2}x<y \\
(x^2,(x-y-z)^2/4,(x-y-z)^2/4,y^2,z^2) & \textit{with } (x,y,z)\in \tilde{P}_0 \\
((x-y+z)^2,(x-y+z)^2,y^2,z^2,z^2+2(z-y+x)^2-y^2) & \textit{with } (x,y,z)\in \tilde{P}_4 \\
((x-y-z)^2/4,(x-y-z)^2/4,x^2,y^2,z^2) & \textit{with } (x,y,z)\in \tilde{\tilde{P}}_0
\end{array}
$$

$$
\begin{array}{ll}
((x-y+z)^2/4, (x-y+z)^2/4, x^2, y^2, z^2) & \textit{with } (x,y,z) \in P_0 \\
((x+y-z)^2/4, (x+y-z)^2/4, x^2, y^2, z^2) & \textit{with } (x,y,z) \in P_0 \\
((x-y+z)^2/9, (x-y+z)^2/9, x^2, y^2, x^2+y^2-2(x-y+z)^2/9) & \textit{with } (x,y,z) \in \tilde{P}_5 \\
((y-x+z)^2/9, (y-x+z)^2/9, x^2, y^2, x^2+y^2-2(y-x+z)^2/9) & \textit{with } (x,y,z) \in \tilde{\tilde{P}}_5 \\
((x-y+z)^2/9, (x-y+z)^2/9, x^2, z^2, x^2+z^2-2(x-y+z)^2/9) & \textit{with } (x,y,z) \in \tilde{P}_6 \ .
\end{array}
$$

We have not examined the five-mode degree five resonances in any detail, although it may present an interesting exercise in the number theory of quadratic forms. The resonance $(1,4,16,25,36)$ is an example with the lowest order, and several others are $(4,9,16,25,36)$, $(4,25,36,49,64)$, and $(1,4,16,100,121)$.

4. The integrable flow

In this section we examine some of the solutions of the integrable degree four normal form. We first consider the invariant subspaces which correspond to energy in only a finite number of modes. We then look at standing wave solutions and travelling wave solutions.

4.1. Invariant subspaces

The four-dimensional (two complex dimensions) linear subspaces $\{(z_k, w_k)\}$ for fixed k, are invariant under the Hamiltonian flow, as are all products of these subspaces. This follows directly from the fact that $\{I_1(k)\}_k$ are integrals. Here, we investigate the flow in the (z_k, w_k)-subspace which represents approximate solutions with one dominant mode in the original water wave problem. For notational simplicity, we will drop the k-subscripts wherever possible.

The motion in the invariant plane is defined by the integrals $I_1(k) = I_1$, $I_2(k) = I_2$ and the time evolution of the angles:

$$
\begin{aligned}
\varphi_1(t) &= \Omega_1 t + \varphi_1(0) = (\sqrt{gk} - \frac{k^3}{4\pi} I_1)t + \varphi_1(0) \\
\varphi_2(t) &= \Omega_2 t + \varphi_2(0) = \frac{3k^3}{4\pi} I_2\, t + \varphi_2(0) \ .
\end{aligned}
$$

A phase portrait of this invariant subspace, restricted to the set $I_1(k) = I_1$ and ignoring the angular evolution of $\varphi_1(t)$ is given in Figure 1. For this figure, we use the integrable coordinates of the linearized problem. The typical solution is a closed curve in the (β, J)-plane. Special solutions are the centers, the vertical boundaries which are all fixed points in the (β, J)-plane, and the horizontal "connecting" orbits.

The surface in the original physical is obtained by applying the inverse normal form transformations to the coordinate functions. It is consistent to approximate z_k and w_k in the original problem with the cubic truncation of the transformations $\exp(\mathrm{ad}_{K_4})(\exp(\mathrm{ad}_{K_3})(z_k))$

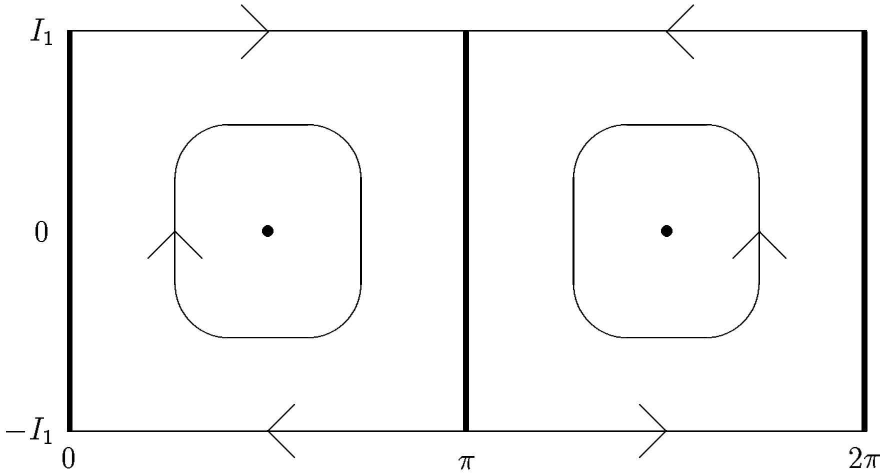

Figure 1: Slice of phase space in (β, J)-plane where $J = (z\bar{z} - w\bar{w})/2$ and $\sin^2\beta = I_2^2/(I_1^2 - J^2)$.

and $\exp(\mathrm{ad}_{K_4})(\exp(\mathrm{ad}_{K_3})(w_k))$, respectively. A one mode solution in the normal form coordinates, of say mode p, corresponds to a solution in the original coordinates of degree three, with three excited modes $p, 2p, 3p$. In our figures we plot the three modes approximation to the surface found using these nonlinear transformations. We used the software package Maple [1] to produce these figures.

The general orbit in this plane has frequencies $\Omega_1(I)$ and $\Omega_2(I)$ which are rationally independent, giving rise to a time quasi-periodic (and of course spatially periodic) solution to the normal forms approximation of the water waves problem. A typical example of the resulting motion of the free surface $\eta(x,t)$ is plotted in Figure 2. Of course whenever the frequency $\Omega_2(I)$ of the 'horizontal drift' of the wave pattern is a rational multiple of the standing frequency $\Omega_1(I)$, this pattern is periodic in time, with typically a rather long period.

There are some distinguished cases which we shall also examine. The first is the standing wave pattern which occurs for motion in the invariant subspace $\{w(k) = 0 : k > 0\}$. In the present coordinates, and restricting the motion to the above two-mode subspace, this corresponds to $I_2 = 0$, which implies that $\Omega_2(I) = 0$. This results in a description of the motion of the free surface in the original physical coordinates involving only the cosine series, of which a solution is plotted in Figure 3. This deep water standing wave was derived by Penny & Price [10], who gave the nonlinear relation between amplitude and frequency for this solution branch.

There is also a family of travelling waves, corresponding to the centers $I_1 = \pm I_2$ in the

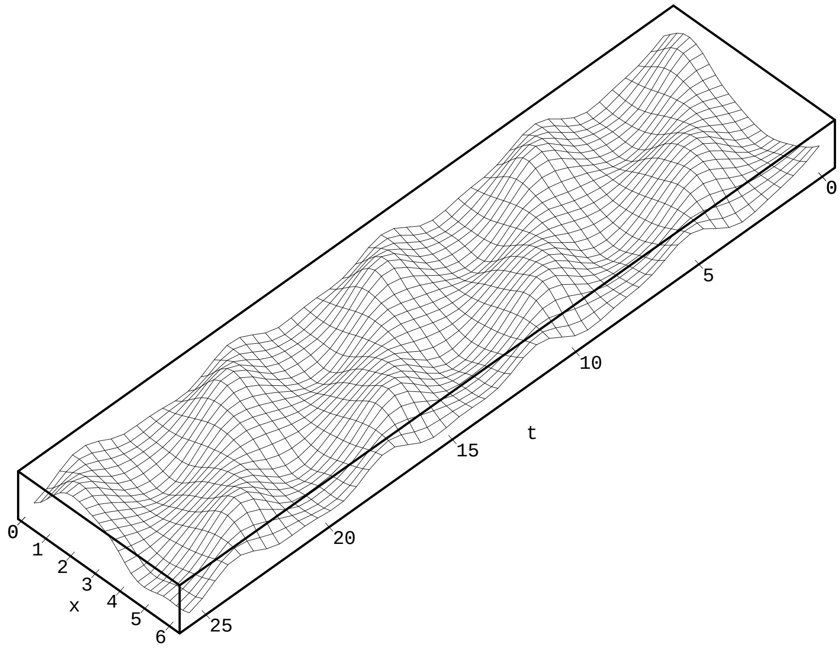

Figure 2: A one mode quasiperiodic wave.

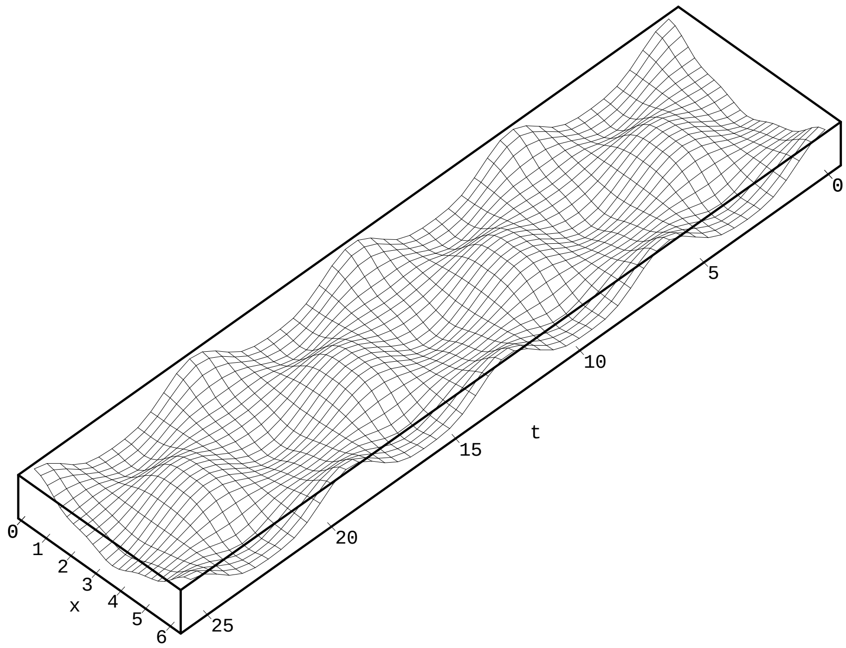

Figure 3: A one mode standing wave.

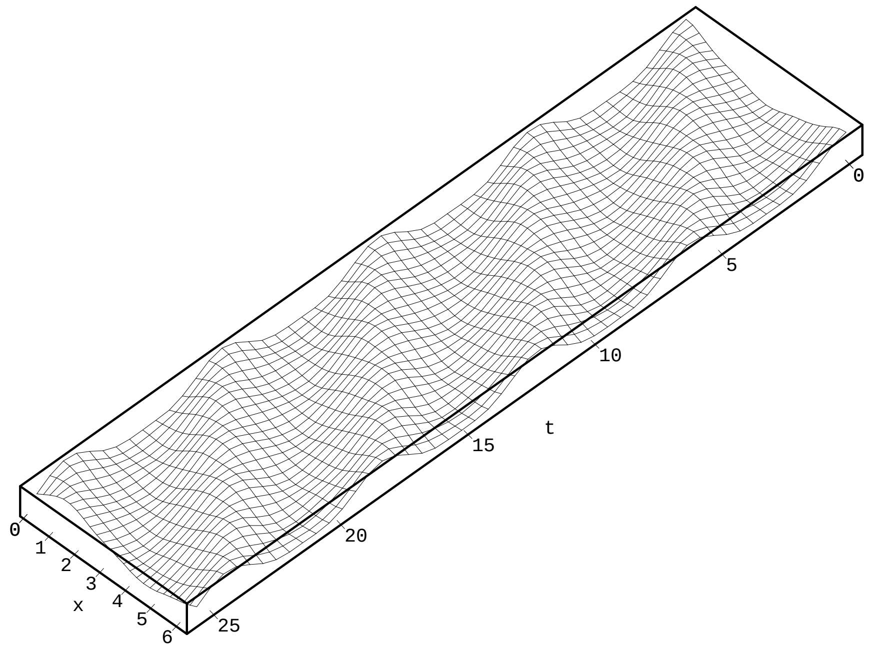

Figure 4: A one mode travelling wave.

phase plane, at which the action-angle coordinate transformation is singular. The resulting solution in physical coordinates is a three modes approximation of the Stokes wave [14], and is pictured in Figure 4. We will discuss standing waves and travelling waves and their bifurcations to higher-mode solutions in more detail in the next two sections.

A final special case is $\Omega_1 = 0$ which occurs when $I_1 = 4\pi\sqrt{gk}/k^3$. The resulting family of solutions are also travelling waves, with speed $c = \Omega_2(I) = 3k^2 I_2/4\pi$. This family of solutions is less physically relevant, since I_1 is not small and therefore is outside of the weakly nonlinear regime in which these approximations are valid.

4.2. Standing wave bifurcations

We now consider standing wave solutions to the normal form. We characterize a standing wave as a periodic wave which satisfies Neumann boundary conditions as given in (2). In our formulation, this reduces to setting all the $w_k \equiv 0$, which simplifies all our expressions and in particular decouples the interaction between modes since $I_2(k) \equiv 0$. The integrable coordinate transformation simplifies and we write

$$z_k(t) = \sqrt{2I_1(k)}e^{-j\varphi_1(t)} .$$

Thus the magnitude of each z_k is fixed and its argument evolves linearly in time. It is now straightforward to look at curves of standing wave solutions with specific properties.

The frequency of each solution mode depends linearly on its energy $I_1(k)$ in the following way

$$\Omega_1(k) = (\sqrt{gk} - \frac{k^3}{4\pi} I_1(k)) \,. \tag{13}$$

We will always assume that the energy $I_1(k)$ is sufficiently small so that this quantity is positive. A standing wave solution consisting of two dominant spatial modes each with the same temporal frequency can occur at the bifurcation points $\Omega_1(k) = \Omega_1(l)$. Such a solution branch bifurcates from a (k)-standing wave when $I_1(k) = \frac{4\pi}{k^3}(\sqrt{gk} - \sqrt{gl})$, and thus requires $l < k$. We can also look at various subharmonic and superharmonic bifurcations. The most interesting cases are the various low frequency solution branches which emerge from the trivial equilibrium surface.

The following propositions are direct consequences of the energy to frequency relationship given in (13). In each of these propositions, we give the condition for a bifurcation from a branch of standing wave solutions with no energy in mode k to a branch with energy in mode k. The original family may be made up of solutions with energies in any number of other modes.

Proposition 9 (harmonic bifurcation) *A branch of standing waves with energy in mode k emanates from a family of standing waves with no energy in mode k in an harmonic bifurcation if the family attains the frequency $\Omega = \sqrt{gk}$. Thus an (l)-standing wave can undergo an harmonic bifurcation to a (k, l)-standing wave when $k < l$ and $I_1(l) = (4\pi/l^3)(\sqrt{gl} - \sqrt{gk})$.*

Proposition 10 (subharmonic bifurcation) *A branch of standing waves with energy in mode k emanates from a family of standing waves with no energy in mode k with period increased by a factor of n in a subharmonic bifurcation if the family attains the frequency $\Omega = n\sqrt{gk}$. Thus an (l)-standing wave can undergo a subharmonic bifurcation to a (k, l)-standing wave with period increased by a factor of n when $n^2 k \leq l$ and $I_1(l) = (4\pi/l^3)(\sqrt{gl} - n\sqrt{gk})$.*

Proposition 11 (superharmonic bifurcation) *A branch of standing waves with energy in mode k emanates from a family of standing waves with no energy in mode k with frequency increased by a factor of n in a superharmonic bifurcation if the family attains the frequency $\Omega = \sqrt{gk}/n$. Hence an (l) standing wave can undergo a superharmonic bifurcation to a (k, l)-standing wave with frequency increased by a factor of n when $k \leq ln^2$ and $I_1(l) = (4\pi/l^3)(\sqrt{gl} - \sqrt{gk}/n)$.*

The bifurcations that are particularly interesting are those which occur at the origin, such as for $n^2 k = l$. One is pictured in Figure 5. In particular, a branch of standing wave solutions including modes $k_1, \ldots, k_m$ can bifurcate from the origin if there are integers $\{n_i\}$ such that $n_1^2 k_1 = n_2^2 k_2 = \ldots = n_m^2 k_m$. The frequency of these standing waves at the bifurcation point is $\sqrt{g n_1^2 k_1}/\text{l.c.m.}(n_1, \ldots, n_m)$, with l.c.m. denoting the least common multiple.

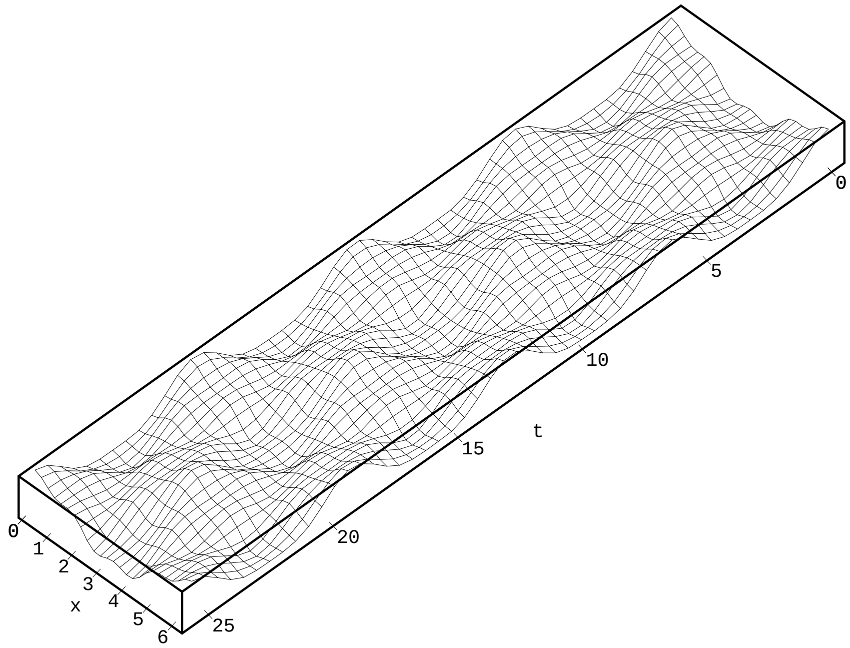

Figure 5: A $(1, 4)$-standing wave.

4.3. Travelling waves

Travelling waves may be characterized as critical points of the momentum on the energy surface, with the phase speed of the wave as Lagrange multiplier; $\nabla(\mathcal{H} - cM) = 0$. Define

$$\nabla_k \equiv \begin{pmatrix} \partial/\partial \bar{z}_k \\ \partial/\partial \bar{w}_k \end{pmatrix} = \begin{pmatrix} z_k & jw_k \\ w_k & -jz_k \end{pmatrix} \begin{pmatrix} \partial/\partial I_1(k) \\ \partial/\partial I_2(k) \end{pmatrix}$$

for functions independent of φ_1 and φ_2. Since $\mathcal{H}$ and M are independent of the angular variables, the condition for a travelling wave is that for all k,

$$\nabla_k(\mathcal{H} - cM) = 0 \ . \tag{14}$$

Using the expressions in (10) and (11), we calculate that

$$\nabla_k(\mathcal{H} - cM) = \frac{1}{2} \begin{pmatrix} z_k & jw_k \\ w_k & -jz_k \end{pmatrix} \begin{pmatrix} \Omega_1(k) \\ \Omega_2(k) - ck \end{pmatrix} \ .$$

We consider two cases, the first when

$$\begin{vmatrix} z_k & jw_k \\ w_k & -jz_k \end{vmatrix} = -j(z_k^2 + w_k^2) = 0$$

and the second when the determinant does not vanish. The trivial solution given by $z_k = w_k = 0$ is of course possible, corresponding to solutions which are not excited in that particular mode.

Case I: $z_k^2 + w_k^2 = 0$

This determinant condition and (14) are satisfied when

$$\begin{aligned} z_k &= \sigma_k j w_k \qquad \Leftrightarrow I_2(k) = \sigma_k I_1(k) \quad (\sigma_k = \pm 1) \\ c &= \sigma_k \frac{\Omega_1(k)}{k} + \frac{\Omega_2(k)}{k} \, . \end{aligned}$$

Case II: $z_k^2 + w_k^2 \neq 0$

This determinant condition and (14) are satisfied when

$$\begin{aligned} z_k &\neq \pm j w_k \\ \Omega_1(k) &= 0 \qquad \Leftrightarrow I_1(k) = \frac{4\pi}{k^3}\sqrt{gk} \\ c &= \Omega_2(k)/k \, . \end{aligned}$$

Labelling a travelling wave by its active modes, *i.e.*, those k for which $I_1(k) \neq 0$, we note that modes excited in the second fashion ($z_k \neq \pm j w_k$) cannot have arbitrarily small amounts of energy. We restrict our attention here to travelling waves with $z_k = \pm j w_k$. It is branches of these types of waves which can bifurcate from the origin in our phase space.

The algebraic conditions from (14) demonstrate that the N modes travelling wave solution is a linear problem in the positive orthant of action variables $\{I_1(k)\}$ corresponding to the active modes. Let

$$J_k = \sigma_k k I_1(k)/\pi \text{ and } d_k = \sigma_k \sqrt{g/k} \, ,$$

and assume that $s(1), \ldots, s(N) \in \mathbf{Z}$ such that $0 < s(1) < \ldots < s(N)$ are the active modes. Then the conditions from (14) reduce to the equation

$$\begin{pmatrix} s(1)/2 & s(1) & s(1) & \cdots & s(1) & -1 \\ s(1) & s(2)/2 & s(2) & \cdots & s(2) & -1 \\ s(1) & s(2) & s(3)/2 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ s(1) & s(2) & \cdots & & s(N)/2 & -1 \end{pmatrix} \begin{pmatrix} J_{s(1)} \\ J_{s(2)} \\ \vdots \\ \vdots \\ J_{s(N)} \\ c \end{pmatrix} = \begin{pmatrix} -d_{s(1)} \\ -d_{s(2)} \\ \vdots \\ \vdots \\ -d_{s(N)} \end{pmatrix} \tag{15}$$

for the $J_{s(i)}$ with the constraint that $\operatorname{sign} J_{s(i)} = \operatorname{sign} d_{s(i)}$. This is a linear equation for $\vec{J}$, with the constraint that the solution must lie in the positive orthant $\{\sigma_{s(i)} J_{s(i)} \geq 0\}$. As the equation depends in an affine manner on $(\vec{J}, c)$, solutions will lie on the intersection of an affine line with this orthant.

A one mode travelling wave of this type was discussed in Section 4.1, and Figure 4 was an example of such a solution in physical coordinates. We will now examine bifurcations on these solution branches to two modes travelling waves. With $k < l$, the conditions from (15) reduce to

$$\begin{aligned} J_k &= 2(d_k - d_l)/k + J_l(2 - l/k) \\ c &= 2d_k - d_l + J_l(2k - l/2) \,, \end{aligned}$$

which is

$$\begin{aligned} I_1(k) &= \tfrac{2\pi}{k^2}\sqrt{\tfrac{g}{k}} - \sigma_k\sigma_l\tfrac{2\pi}{k^2}\sqrt{\tfrac{g}{l}} + \sigma_k\sigma_l\tfrac{l}{k}(2 - l/k)I_1(l) \\ c &= 2\sigma_k\sqrt{\tfrac{g}{k}} - \sigma_l\sqrt{\tfrac{g}{l}} + \sigma_l\tfrac{l}{\pi}(2k - l/2)I_1(l) \,. \end{aligned} \tag{16}$$

We examine this for arbitrary (k, l), for two cases, the first is $\sigma_k = \sigma_l = 1$ and the second is $\sigma_k = -\sigma_l = 1$ (the other two possibilities give the same solutions with opposite wave speed c). The following propositions describe the bifurcation of these branches of solutions from branches of one mode travelling wave solutions. We call the bifurcation supercritical if the wave speed, $|c|$, increases along the two modes branch and subcritical if it decreases.

Proposition 12 *A branch of (k,l)-travelling wave solutions with $k < l$ and $\sigma_k = \sigma_l = 1$ bifurcates from a branch of (k)-travelling wave solutions at $I_1(k) = (2\pi/k^2)(\sqrt{g/k} - \sqrt{g/l})$. The bifurcation is supercritical if $k > l/4$ and subcritical if $k < l/4$.*

Proposition 13 *A branch of (k,l)-travelling wave solutions with $k < l$ and $\sigma_k = \sigma_l = 1$ bifurcates from a branch of (l)-travelling wave solutions if $k < l/2$. The bifurcation occurs at $I_1(l) = (2\pi/l(l-2k))\left(\sqrt{g/k} - \sqrt{g/l}\right)$ and is supercritical if $k < l/4$ and subcritical if $k > l/4$.*

Proposition 14 *A branch of (k,l)-travelling wave solutions with $k < l$ and $\sigma_k = -\sigma_l = 1$ bifurcates from a branch of (k)-travelling wave solutions at $I_1(k) = (2\pi/k^2)(\sqrt{g/k} + \sqrt{g/l})$. The bifurcation is supercritical if $k < l/4$ and subcritical if $k > l/4$.*

Proposition 15 *A branch of (k,l)-travelling wave solutions with $k < l$ and $\sigma_k = -\sigma_l = 1$ bifurcates from a branch of (l)-travelling wave solutions if $k > l/2$. The bifurcation occurs at $I_1(l) = (2\pi/l(2k-l))\left(\sqrt{g/k} + \sqrt{g/l}\right)$ and is subcritical.*

These propositions are all consequences of (16). Using Propositions 12 and 13, in Figure 6 we sketch the bifurcation diagrams for one type of (k)-travelling wave, (l)-travelling wave, and (k,l)-travelling wave for the cases $k < l \le 2k$ and $2k < l$.

We now make some statements about general N modes travelling waves. Define $M(s)$ to be the matrix from (15).

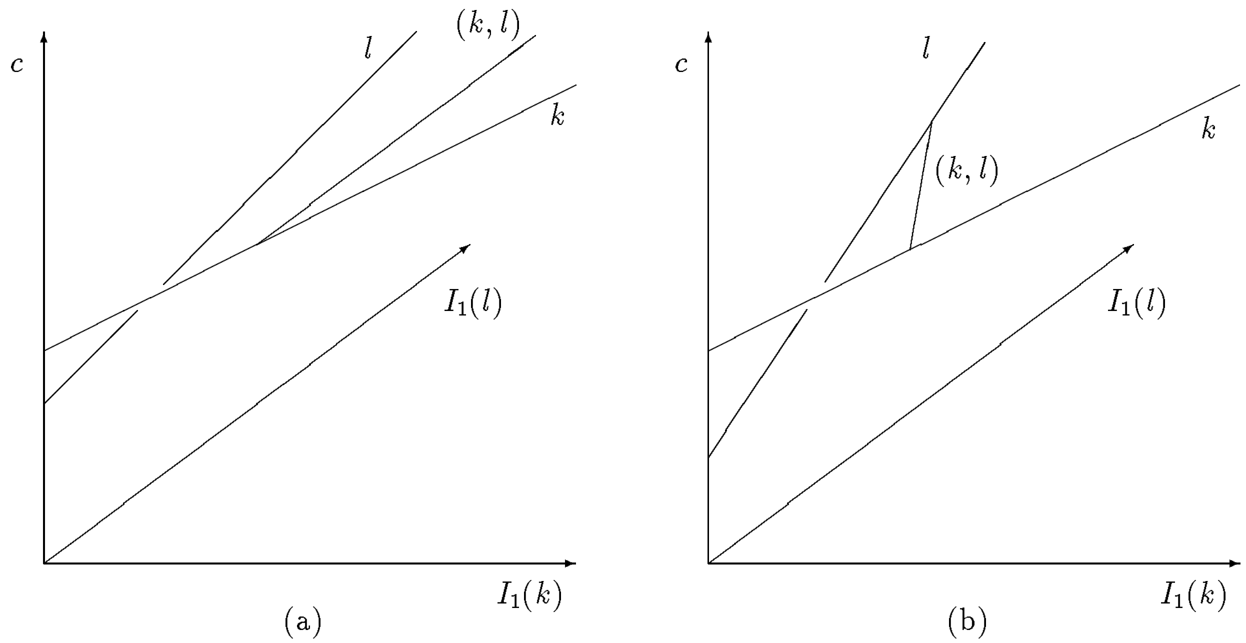

Figure 6: Bifurcation diagram for (k)-travelling wave, (l)-travelling wave, and (k,l)-travelling wave when: (a) $k < l < 2k$, (b) $2k < l$.

Lemma 16 *Rank* $(M(s)) = N$.

This is easily proven by performing column operations on $M(s)$. Another immediate consequence of these column operations is contained in the following lemma.

Lemma 17 *The Nth column of $M(s)$ is a linear combination of the other columns.*

These lemmas provide the proof for the following proposition regarding solutions to the travelling wave solution.

Proposition 18 *The solutions to Equation 15, which we write as $M(s)\mathcal{J} = \mathcal{D}$, can always be written in the form*

$$\mathcal{J} = \mathcal{J}_P + J_{s(N)}\mathcal{J}_H$$

for some fixed $\mathcal{J}_P$ and $\mathcal{J}_H$.

Essentially, this means that a branch of solutions can be parameterized by the energy in the highest mode. Thus travelling wave solutions occur in families consisting of intervals $L_N = (l_m, l_M)$, possibly infinite, such that each $J_{s(N)} \in L_N$ gives a solution. There will be a corresponding interval C_N for which $c(J_{s(N)}) \in C_N$ which is either open (possibly infinite) or consists of only one point.

Proposition 19 *Assume that a family of N modes travelling wave solutions exists and is given by $\mathcal{J} = \mathcal{J}_P + J_{s(N)}\mathcal{J}_H$ for $J_{s(N)} \in L_N = (l_m, l_M)$. We consider the bifurcation of a branch of $N+1$ modes solutions with new mode $j \neq s(i)$ and $\sigma_j = \pm 1$ from this family of solutions. Define the vector $\mathcal{V}$ by*

$$\mathcal{V}(i) = \begin{cases} s(i), \text{ if } 1 \leq i \leq N \text{ and } s(i) < j \\ j, \text{ if } 1 \leq i \leq N \text{ and } s(i) > j \\ -1, i = N+1 \end{cases} .$$

This new branch of solutions exists if and only if

$$d_j \in \left(\mathcal{V}^T \mathcal{J}_P + l_m \mathcal{V}^T \mathcal{J}_{\mathcal{H}},\ \mathcal{V}^T \mathcal{J}_P + l_M \mathcal{V}^T \mathcal{J}_{\mathcal{H}}\right) .$$

This proposition follows from considering solutions to (15) in dimensions N and N+1.

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